Transverse foliations on the torus T2 and partially hyperbolic diffeomorphisms on 3-manifolds

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Transverse foliations on the torus $\mathbb{T}^2$ and partially hyperbolic diffeomorphisms on 3-manifolds

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Abstract. In this paper, we prove that given two C^1 foliations $\mathcal{F}$ and $\mathcal{G}$ on $\mathbb{T}^2$ which are transverse, there exists a non-null homotopic loop $\{\Phi_t\}_{t\in[0,1]}$ in $\mathrm{Diff}^1(\mathbb{T}^2)$ such that $\Phi_t(\mathcal{F}) \pitchfork \mathcal{G}$ for every $t \in [0,1]$, and $\Phi_0 = \Phi_1 = \mathrm{Id}$.

As a direct consequence, we get a general process for building new partially hyperbolic diffeomorphisms on closed 3-manifolds. Bonatti et al. [4] built a new example of dynamically coherent non-transitive partially hyperbolic diffeomorphism on a closed 3-manifold; the example in [4] is obtained by composing the time t map, $t > 0$ large enough, of a very specific non-transitive Anosov flow by a Dehn twist along a transverse torus. Our result shows that the same construction holds starting with any non-transitive Anosov flow on an oriented 3-manifold. Moreover, for a given transverse torus, our result explains which type of Dehn twists lead to partially hyperbolic diffeomorphisms.

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1. Introduction and statement of the main results

The main motivation of this paper is the construction of new examples of partially hyperbolic diffeomorphisms on closed 3-manifolds, initiated in [4]. More precisely, our main result is a topological result which was missing for [4] getting a general construction instead of a precise example. Nevertheless, this topological result deals with very elementary objects and is interesting by itself. We first present it below independently from its application on partially hyperbolic diffeomorphisms.

1.1. Pair of transverse foliations on $\mathbb{T}^2$. The space of 1-dimensional (non-singular) smooth foliations on the torus $\mathbb{T}^2$ has several connected components which are easy to describe: such a foliation is directed by a smooth line field, which can be seen as a map $X : \mathbb{T}^2 \to \mathbb{RP}^1 \simeq \mathbb{S}^1$; such a map induces a morphism $X_* : \pi_1(\mathbb{T}^2) = \mathbb{Z}^2 \to \mathbb{Z}$ and two foliations can be joined by a path of non-singular foliations if and only if the induced morphisms coincide. The group $\mathrm{Diff}_0(\mathbb{T}^2)$ of diffeomorphisms of $\mathbb{T}^2$ isotopic to the identity map has a natural action on the space of foliations.

In this paper, we consider pairs $(\mathcal{F}, \mathcal{G})$ of transverse foliations on $\mathbb{T}^2$. For any such a pair $(\mathcal{F}, \mathcal{G})$ of transverse foliations, we consider the open subset of $\mathrm{Diff}_0(\mathbb{T}^2)$ of all diffeomorphism φ so that $\varphi(\mathcal{F})$ is transverse to $\mathcal{G}$. Our main result below shows that this open subset contains non-trivial loops.

Theorem A. *Let $\mathcal{F}$ and $\mathcal{G}$ be two C^1 one-dimensional foliations on $\mathbb{T}^2$ and they are transverse. Then there exists a continuous (for the C^1-topology) family $\{\Phi_t\}_{t\in[0,1]}$ of C^1 diffeomorphisms on $\mathbb{T}^2$ such that*

- $\Phi_0 = \Phi_1 = \mathrm{Id}$*;*
- *For every $t \in [0,1]$, the C^1 foliation $\Phi_t(\mathcal{F})$ is transverse to $\mathcal{G}$;*
- *For every point $x \in \mathbb{T}^2$, the closed curve $\Phi_t(x)$ is non-null homotopic.*

Our main theorem is implied by the following two theorems, according to the two cases described in Definition 1.1 below:

Definition 1.1. We say that two foliations $\mathcal{F}$ and $\mathcal{G}$ of the torus $\mathbb{T}^2$ *have parallel compact leaves* if and only if there exist a compact leaf of $\mathcal{F}$ and a compact leaf of $\mathcal{G}$ which are in the same free homotopy class.

Otherwise, we say that $\mathcal{F}$ and $\mathcal{G}$ *have no parallel compact leaves* or that they are *without parallel compact leaves.*

Theorem 1.2. *Let $\mathcal{F}$ and $\mathcal{G}$ be two C^1 one-dimensional transverse foliations on $\mathbb{T}^2$, without parallel compact leaves. Then for any $\alpha \in \pi_1(\mathbb{T}^2)$, there exists a C^1-continuous family $\{\Phi_t\}_{t\in[0,1]}$ of C^1 diffeomorphisms on $\mathbb{T}^2$ such that*

- $\Phi_0 = \Phi_1 = \mathrm{Id}$*;*
- *For every $t \in [0,1]$, the C^1 foliation $\Phi_t(\mathcal{F})$ is transverse to $\mathcal{G}$;*
- *For every point $x \in \mathbb{T}^2$, the closed curve $\Phi_t(x)$ is in the homotopy class of α.*

The proof of Theorem 1.2 consists in endowing $\mathbb{T}^2$ with coordinates in which the foliations $\mathcal{F}$ and $\mathcal{G}$ are separated by 2 affine foliations (i.e. $\mathcal{F}$ and $\mathcal{G}$ are tangent to two transverse constant cones). Thus in these coordinates every translation leaves $\mathcal{F}$ transverse to $\mathcal{G}$, concluding.

Theorem 1.3. *Let $\mathcal{F}$ and $\mathcal{G}$ be two C^1 one-dimensional foliations on $\mathbb{T}^2$ and they are transverse. Assume that $\mathcal{F}$ and $\mathcal{G}$ have parallel compact leaves which are in the homotopy class $\alpha \in \pi_1(\mathbb{T}^2)$. Then, for each $\beta \in \pi_1(\mathbb{T}^2)$, one has that $\beta \in \langle\alpha\rangle$ if and only if there exists a C^1-continuous family $\{\Phi_t\}_{t\in[0,1]}$ of C^1 diffeomorphisms on $\mathbb{T}^2$ such that*

- $\Phi_0 = \Phi_1 = \mathrm{Id}$*;*
- *For every $t \in [0,1]$, the C^1 foliation $\Phi_t(\mathcal{F})$ is transverse to $\mathcal{G}$;*
- *For every point $x \in \mathbb{T}^2$, the closed curve $\Phi_t(x)$ is in the homotopy class of β.*

One easily checks that, if $\mathcal{F}$ and $\mathcal{G}$ are transverse C^1 foliations having compact leaves in the same homotopy class, then every compact leaf $L_{\mathcal{F}}$ of $\mathcal{F}$ is disjoint from every compact leaf $L_{\mathcal{G}}$ of $\mathcal{G}$. If $\{\Phi_t\}_{t\in[0,1]}$ is an isotopy so that Φ_0 is the identity map and $\Phi_t(\mathcal{F})$ is transverse to $\mathcal{G}$, then $\Phi_t(L_{\mathcal{F}})$ remains disjoint from $L_{\mathcal{G}}$: this implies the *if* part of Theorem 1.3. The *only if* part will be the aim of Section 6. The proof consists in endowing $\mathbb{T}^2$ with coordinates in which $\mathbb{T}^2$ is divided into vertical adjacent annuli in which the foliations are separated by affine foliations: now the vertical translations preserve the vertical annuli and map $\mathcal{F}$ on foliations transverse to $\mathcal{G}$.

Remark 1.4. First notice that every continuous path (for the C^1-topology) of C^1 diffeomorphisms can be approached, in the C^1-topology, by a smooth path of smooth diffeomorphisms.

Now, as the transversality of foliations is an open condition, any loop $\{\Psi_t\}_{t\in[0,1]}$ of diffeomorphisms C^1-close to the loop $\{\Phi_t\}_{t\in[0,1]}$ (announced in Theorems A, 1.2, and 1.3) satisfies that $\Psi_t(\mathcal{F})$ is transverse to $\mathcal{G}$ for every t.

Therefore, in Theorems A, 1.2, and 1.3, one can choose the loop $t \mapsto \Phi_t$ so that the map $(t,x) \mapsto \Phi_t(x)$, for $(t,x) \in \mathbb{S}^1 \times \mathbb{T}^2$, is smooth.

Definition 1.5. Let $(\mathcal{F},\mathcal{G})$ be a pair of transverse foliations of $\mathbb{T}^2$. We denote by $G_{\mathcal{F},\mathcal{G}} \subset \pi_1(\mathbb{T}^2)$ the group defined as follows:

- if $\mathcal{F}$ and $\mathcal{G}$ have no parallel compact leaves, then $G_{\mathcal{F},\mathcal{G}} = \mathbb{Z}^2 = \pi_1(\mathbb{T}^2)$;
- if $\mathcal{F}$ and $\mathcal{G}$ have parallel compact leaves, let $\alpha \in \pi_1(\mathbb{Z})$ be the homotopy class of these leaves. Then $G_{\mathcal{F},\mathcal{G}} = \langle\alpha\rangle = \mathbb{Z}\cdot\alpha \subset \pi_1(\mathbb{T}^2)$.

1.2. Dehn twists and pairs of transverse 2-foliations on 3-manifolds.

The aim of this paper is to build partially hyperbolic diffeomorphisms on 3-manifolds by composing the time t-map of an Anosov flow by a Dehn twist along a transverse tori. In this section we define the notion of Dehn twists, and we state a straightforward consequence of Theorems 1.2 and 1.3 producing Dehn twists preserving the transversality of two 2-dimensional foliations.

Definition 1.6. Let $u = (n,m) \in \mathbb{Z}^2 = \pi_1(\mathbb{T}^2)$. A diffeomorphism $\psi : [0,1] \times \mathbb{T}^2 \to [0,1] \times \mathbb{T}^2$ is called a *Dehn twist of* $[0,1] \times \mathbb{T}^2$ *directed by* u if:

- ψ is of the form $(t,x) \mapsto (t,\psi_t(x))$, where ψ_t is a diffeomorphism of $\mathbb{T}^2$ depending smoothly on t.
- ψ_t is the identity map for t close to 0 or close to 1.
- the closed path $t \mapsto \psi_t(O)$ on $\mathbb{T}^2$ is freely homotopic to u (where $O = (0,0)$ in $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$).

Definition 1.7. Let M be an oriented 3-manifold and let $T : \mathbb{T}^2 \hookrightarrow M$ be an embedded torus. Fix $u \in \pi_1(T)$. We say that a diffeomorphism $\psi : M \to M$ is *a*

Dehn twist along T directed by u if there is an orientation preserving diffeomorphism $\varphi : [0,1] \times \mathbb{T}^2 \hookrightarrow M$ whose restriction to $\{0\} \times \mathbb{T}^2$ induces T, and so that:

- ψ is the identity map out of $\varphi([0,1] \times \mathbb{T}^2)$. In particular, ψ leaves invariant $\varphi([0,1] \times \mathbb{T}^2)$;
- The diffeomorphism $\varphi^{-1} \circ \psi \circ \varphi : [0,1] \times \mathbb{T}^2 \to [0,1] \times \mathbb{T}^2$ is a Dehn twist directed by u.

Proposition 1.8. *Let $\mathcal{F}$, $\mathcal{G}$ be a pair of 2-dimensional foliations on a 3-manifold M, and let $\mathcal{E}$ be the 1-dimensional foliation obtained as $\mathcal{E} = \mathcal{F} \cap \mathcal{G}$. Assume that there is an embedded torus $T \subset M$ which is transverse to $\mathcal{E}$ (hence T is transverse to $\mathcal{F}$ and $\mathcal{G}$). We denote by $\mathcal{F}_T$, $\mathcal{G}_T$ the 1-dimensional foliations on T obtained as intersection of $\mathcal{F}$ and $\mathcal{G}$ with T, respectively.*

Then for every $u \in G_{\mathcal{F}_T,\mathcal{G}_T} \subset \pi_1(T)$, there is a Dehn twist ψ along T directed by u so that $\psi(\mathcal{F})$ is transverse to $\mathcal{G}$.

1.3. Building partially hyperbolic diffeomorphisms on 3-manifolds. In order to state our main result, we first need to define the notions of *partially hyperbolic diffeomorphism* and of *Anosov flow*.

1.3.1. Definition of partially hyperbolic diffeomorphisms. A diffeomorphism f of a Riemannian closed 3-manifold M is called *partially hyperbolic* if there is a Df-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ in direct sum of 1-dimensional bundles so that

(P1) There is an integer $N > 0$ such that for any $x \in M$ and any unit vectors $u \in E^s(x)$, $v \in E^c(x)$ and $w \in E^u(x)$, one has:

$$\|Df^N(u)\| < \inf\{1, \|Df^N(v)\|\} \leq \sup\{1, \|Df^N(v)\|\} < \|Df^N(w)\|.$$

A diffeomorphism f of a Riemannian closed 3-manifold M is called *absolute partially hyperbolic* if it is partially hyperbolic satisfying the stronger assumption

(P2) There are $0 < \lambda < 1 < \sigma$ and an integer $N > 0$ so that for any $x, y, z \in M$ and any unit vectors $u \in E^s(x)$, $v \in E^c(y)$ and $w \in E^u(z)$, one has:

$$\|Df^N(u)\| < \lambda < \|Df^N(v)\| < \sigma < \|Df^N(w)\|.$$

We refer the readers to [2, Appendix B] for the first elementary properties and [17] for a survey book of results and questions for partially hyperbolic diffeomorphisms.

1.3.2. Definition of Anosov flows. A vector field X on a 3-manifold M is called an *Anosov vector field* if there is a splitting $TM = E^s \oplus \mathbb{R} \cdot X \oplus E^u$ as a direct sum of 1-dimensional bundles which are invariant by the flow $\{X_t\}_{t \in \mathbb{R}}$ of X, and so that the vectors in E^s are uniformly contracted and the vectors in E^u are uniformly expanded by the flow of X.

Notice that X is Anosov if and only if X has no zeros and if there is $t > 0$ so that X_t is partially hyperbolic.

The bundles $E^{cs} = E^s \oplus \mathbb{R} \cdot X$ and $E^{cu} = \mathbb{R} \cdot X \oplus E^u$ are called the weak stable and unstable bundles (respectively). They are tangent to transverse 2-dimensional foliations denoted by $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ respectively, which are of class C^1 if X is of class at least C^2.

The bundles E^s and E^u are called the strong stable and strong unstable bundles, and are tangent to 1-dimensional foliations denoted by $\mathcal{F}^{ss}$ and $\mathcal{F}^{uu}$ which are called the strong stable and the strong unstable foliations, respectively.

Notice that being an Anosov vector field is an open condition in the set of C^1-vector fields and that the structural stability implies that all the flows C^1-close to an Anosov flow are Anosov flows topologically equivalent to it. Therefore, for our purpose here we may always assume, and we do it, that the Anosov flows we consider are smooth.

The most classical Anosov flows on 3-manifolds are the geodesic flows of hyperbolic closed surfaces and the suspension of hyperbolic linear automorphisms of $\mathbb{T}^2$ (i.e. induced by an hyperbolic element of $SL(2, \mathbb{Z})$). In 1979, [8] built the first example of a non-transitive Anosov flow on a closed 3-manifold. Many other examples of transitive or non-transitive Anosov flows have been built in [1,3].

1.3.3. Transverse tori. If X is an Anosov vector field on an oriented closed 3-manifold M and if $S \subset M$ is an immersed closed surface which is transverse to X then

- S is oriented (as transversely oriented by X);
- S is transverse to the weak foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ of X and these foliations induce on S two 1-dimensional C^1-foliations $\mathcal{F}^s_S$ and $\mathcal{F}^u_S$, respectively, which are transverse.
- as a consequence of the two previous items, S is a torus.

A *transverse torus* is an embedded torus $T : \mathbb{T}^2 \hookrightarrow M$ transverse to X and we denote by $\mathcal{F}^s_T$ and $\mathcal{F}^u_T$ the 1-dimensional C^1 foliations induced on T obtained by intersections of T with $\mathcal{F}^{cs}$ and with $\mathcal{F}^{cu}$, respectively. These foliations are transverse. Therefore Theorem A associates to $(\mathcal{F}^s_T, \mathcal{F}^u_T)$ and a subgroup $G_{\mathcal{F}^s_T, \mathcal{F}^u_T}$ of $\pi_1(T)$ which is either a cyclic group if $\mathcal{F}^s_T$ and $\mathcal{F}^u_T$ have parallel compact leaves or the whole $\pi_1(T)$ otherwise.

Let $T_1, \dots, T_k$ be a finite family of transverse tori. We say that X *has no return on* $\bigcup_i T_i$ if each torus T_i is an embedded torus, the $\{T_i\}$ are pairwise disjoint and each orbit of X intersects $\bigcup_i T_i$ in at most 1 point.

A *Lyapunov function* for X is a function which is not increasing along every orbit, and which is strictly decreasing along every orbit which is not chain recurrent.

In [5] Marco Brunella noticed that a non-transitive Anosov vector field X on an oriented closed 3-manifold M always admits a smooth Lyapunov function whose regular levels separate the *basic pieces* of the flow; such a regular level is a disjoint union of transverse tori $T_1, \dots, T_k$. One can check the following statement:

Proposition 1.9. *Let X be a (non-transitive) Anosov vector field on an oriented closed 3-manifold M. Then the two following assertions are equivalent:*

(1) *$T_1, \dots, T_k$ are transverse tori so that X has no return on $\bigcup_i T_i$.*

(2) *there is a smooth Lyapunov function $\theta : M \to \mathbb{R}$ of X for which the T_i, $i \in \{1, \dots, k\}$ are (distinct) connected components of the same regular level $\theta^{-1}(t)$ for some $t \in \mathbb{R}$.*

We are now ready to state our main result.

1.4. Statement of our main result.

Theorem B. *Let X be a smooth (non-transitive) Anosov vector field on an oriented closed 3-manifold M, and let $T_1, \dots, T_k$ be transverse tori so that X has no return on $\bigcup_i T_i$. We endow each T_i with the pair $(\mathcal{F}_i^s, \mathcal{F}_i^u)$ of transverse 1-dimensional C^1 foliations obtained by intersections of T_i with the weak stable and unstable (respectively) foliations of X; let*

$$G_i = G_{\mathcal{F}_i^s, \mathcal{F}_i^u} \subset \pi_1(T_i)$$

denote the subgroup associated to the pair $(\mathcal{F}_i^s, \mathcal{F}_i^u)$ by Theorems 1.2 and 1.3.

Then for any family $u_1 \in G_1, \dots, u_k \in G_k$ there is a family Ψ_i of Dehn twists along T_i directed by u_i, and whose supports are pairwise disjoint and there is $t > 0$ so that the composition

$$f = \Psi_1 \circ \Psi_2 \circ \cdots \circ \Psi_k \circ X_t$$

is an absolute partially hyperbolic diffeomorphism of M.

Furthermore, f is robustly dynamically coherent, the center stable foliation $\mathcal{F}_f^{cs}$ and center unstable foliation $\mathcal{F}_f^{cu}$ are plaque expansive.

In a forthcoming work with a different group of authors, one will remove the hypothesis that the T_i are connected components of a regular level of a Lyapunov function. We state this result here with this restrictive hypothesis in order that this result is a straightforward consequence of Theorems 1.2 and 1.3. Removing this hypothesis will require further very different arguments.

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2. Preliminary: foliations on the torus and its classification

In this section, we give the definitions and results we need. We denote by $\mathbb{S}^1$ the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and by $\mathbb{T}^2$ the torus $\mathbb{R}^2/\mathbb{Z}^2 = \mathbb{S}^1 \times \mathbb{S}^1$.

2.1. Complete transversal.

Definition 2.1. Given a C^r $(r \geq 1)$ foliation $\mathcal{F}$ on $\mathbb{T}^2$. We say that a C^1 simple closed curve γ is *a complete transversal* or a *complete transverse cross section* of the foliation $\mathcal{F}$, if γ is transverse to $\mathcal{F}$ and every leaf of $\mathcal{F}$ intersects γ.

Lemma 2.2. *Consider a C^1 foliation $\mathcal{E}$ on $\mathbb{T}^2$. Assume that there is a simple smooth closed curve γ which is transverse to $\mathcal{E}$ and is not a complete transversal of $\mathcal{E}$. Then there exists a compact leaf of $\mathcal{E}$ which is in the homotopy class of γ.*

This lemma is classical. As the proof is short, we include it for completeness.

Proof. Cut the torus along γ: we get a cylinder C endowed with a foliation transverse to its boundary. Furthermore, by assumption, this foliation admits a leaf which remains at a uniform distance away from the boundary of C.

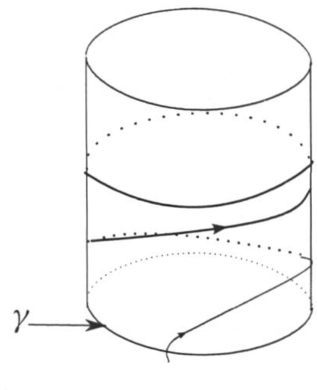

Figure 1.

Hence, the closure of that leaf is also far from the boundary of C. By the Poincaré–Bendixson theorem, the closure of this leaf contains a compact leaf, thus this compact leaf is disjoint from the boundary C. Furthermore, as the foliation is not singular, this leaf cannot be homotopic to 0 in the annulus, hence it is homotopic to the boundary components. □

Definition 2.3. Given two C^r foliations $\mathcal{F}$ and $\mathcal{F}'$ on manifolds M and M', respectively. $\mathcal{F}$ and $\mathcal{F}'$ are C^r *conjugate* if there exists a C^r diffeomorphism $f : M \to M'$ such that $f(\mathcal{F}) = \mathcal{F}'$.

2.2. Reeb components.

Definition 2.4 (Reeb component). Given a foliation $\mathcal{F}$ on $\mathbb{T}^2$, we say that $\mathcal{F}$ has a *Reeb component*, if there exists a compact annulus A such that

- the boundary ∂A is the union of two compact leaves of $\mathcal{F}$;
- there is no compact leaf in the interior of A;
- first item above implies that $\mathcal{F}$ is orientable restricted to A, so let us choose an orientation. We require that the two oriented compact leaves are in opposite homotopy classes.

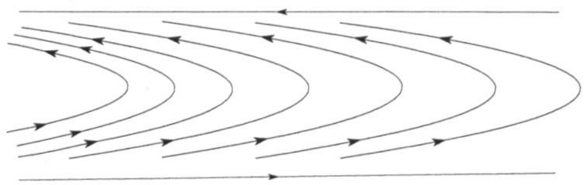

Figure 2. Reeb component.

By using the Poincaré–Bendixson theorem, one easily checks the following classical result:

Proposition 2.5. *Let $\mathcal{F}$ and $\mathcal{G}$ be two transverse foliations on $\mathbb{T}^2$. Assume that $\mathcal{F}$ admits a Reeb component A. Then $\mathcal{G}$ admits a compact leaf contained in the interior of A. Thus $\mathcal{F}$ and $\mathcal{G}$ have parallel compact leaves.*

We state now a classification theorem which can be found in [13]:

Theorem 2.6 ([13, Proposition 4.3.2]). *For any C^r foliation $\mathcal{F}$ on $\mathbb{T}^2$, we have the following:*

- *Either $\mathcal{F}$ has Reeb component; or*
- *$\mathcal{F}$ is C^r conjugated to the suspension of a C^r diffeomorphism on S^1.*

In general the union of the compact leaves of a foliation may fail to be compact. But, for codimension 1 foliations we have the following theorem due to A. Haefliger.

Theorem 2.7 ([10]). *For any C^r $(r \geq 1)$ codimension one foliation $\mathcal{F}$ on a compact manifold M, the set*

$$\{x \in M \mid \textit{The } \mathcal{F}\textit{-leaf through } x \textit{ is compact}\}$$

is a compact subset of M.

2.3. Translation and rotation numbers. In this section we recall very classical Poincaré theory on the rotation number of a circle homeomorphism. We refer to [11] for more details.

We denote by $\widetilde{\text{homeo}}^+(\mathbb{R})$ the set of orientation preserving homeomorphisms on $\mathbb{R}$ which commute with the translation $t \mapsto t+1$. Recall that the elements of $\widetilde{\text{homeo}}^+(\mathbb{R})$ are precisely the lifts on $\mathbb{R}$ of the orientation preserving homeomorphisms of $\mathbb{S}^1$.

Let $H \in \widetilde{\text{homeo}}^+(\mathbb{R})$ be the lift of $h \in \text{Homeo}^+(\mathbb{S}^1)$. Poincaré noticed that the ratio $\frac{H^n(x)-x}{n}$ converges uniformly, as $n \to \pm\infty$, to some constant $\tau(H)$ called the *translation number of* H. The projection $\rho(h)$ of $\tau(H)$ on $\mathbb{R}/\mathbb{Z}$ does not depend on the lift H and is called *the rotation number of* h.

We can find the following observations in many books, in particular in [11].

Remark 2.8. The rotation number is rational if and only if h admits a periodic point.

Proposition 2.9 ([11, Proposition 11.1.6]). $\tau(\cdot)$ *varies continuous in* C^0*-topology.*

Proposition 2.10 ([11, Proposition 11.1.9]). *Let* $H, F \in \widetilde{\text{homeo}}^+(\mathbb{R})$. *Assume that* $\tau(H)$ *is irrational and* $H(x) < F(x)$, *for any* $x \in \mathbb{R}$. *Then we have that* $\tau(H) < \tau(F)$.

Poincaré theory proves that a homeomorphism h of the circle with irrational rotation number is semi-conjugated to the rotation $R_{\rho(h)}$; but h may fail to be conjugated to $R_{\rho(h)}$ even if h is a C^1-diffeomorphism (Denjoy counter examples). However the semi-conjugacy is a C^0-conjugacy if h is a C^2-diffeomorphism (Denjoy theorem). In general, if h is a smooth circle diffeomorphism with irrational rotation number, the conjugacy to the corresponding rotation may fail to be a diffeomorphism. However M. Herman proved that there are generic conditions on $\rho(h)$ ensuring the smoothness of the conjugacy:

Theorem 2.11 ([14]). *Let* $f \in \text{Diff}^r(\mathbb{S}^1)$ $(r \geq 3)$ *be a diffeomorphism of the circle. If the rotation number of* f *is diophantine, then* f *is* C^{r-2} *conjugated to an irrational rotation.*

2.4. Foliations without compact leaves on the annulus. Let $\mathcal{F}$ be a C^r foliation on the annulus $\mathbb{S}^1 \times [0,1]$ so that

- $\mathcal{F}$ is transverse to the boundary $\mathbb{S}^1 \times \{0,1\}$;
- $\mathcal{F}$ has no compact leaves in the annulus.

Thus Poincaré–Bendixson theorem implies that every leaf entering through $\mathbb{S}^1 \times \{0\}$ goes out through a point of $\mathbb{S}^1 \times \{1\}$. The map $\mathcal{P}_{\mathcal{F}} : \mathbb{S}^1 \times \{0\} \to \mathbb{S}^1 \times \{1\}$, which associates an entrance point $(x,0)$ of a leaf in the annulus to its exit point at $\mathbb{S}^1 \times \{1\}$, is called the holonomy of $\mathcal{F}$.

Consider the universal cover $\mathbb{R} \times [0,1] \to \mathbb{S}^1 \times [0,1]$; we denote by $\widetilde{\mathcal{F}}$ the lift of $\mathcal{F}$ on $\mathbb{R} \times [0,1]$ and by $\widetilde{\mathcal{P}_{\mathcal{F}}}$ the holonomy of $\widetilde{\mathcal{F}}$. Note that $\widetilde{\mathcal{P}_{\mathcal{F}}}$ is a lift of $\mathcal{P}_{\mathcal{F}}$.

We will use the following classical and elementary results:

Proposition 2.12. *Let $\mathcal{F}, \mathcal{G}$ be C^r-foliations, $r \geq 0$, on the annulus $\mathbb{S}^1 \times [0,1]$ so that:*

- *The foliations $\mathcal{F}$ and $\mathcal{G}$ are transverse to the boundary $\mathbb{S}^1 \times \{0,1\}$ and have no compact leaves in the annulus;*
- *The foliations $\mathcal{F}$ and $\mathcal{G}$ coincide in a neighborhood of the boundary $\mathbb{S}^1 \times \{0,1\}$;*
- *The foliations $\mathcal{F}$ and $\mathcal{G}$ have same holonomy, that is $\mathcal{P}_{\mathcal{F}} = \mathcal{P}_{\mathcal{G}}$.*

Then there is a C^r diffeomorphism $\varphi : \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1 \times [0,1]$ which coincides with the identity map in a neighborhood of the boundary $\mathbb{S}^1 \times \{0,1\}$ and so that

$$\varphi(\mathcal{G}) = \mathcal{F}.$$

If furthermore the lifted foliations $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ have same holonomies, that is $\widetilde{\mathcal{P}_{\mathcal{F}}} = \widetilde{\mathcal{P}_{\mathcal{G}}}$, then φ is isotopic (relative to a neighborhood of the boundary) to the identity map.

An important step for proving Proposition 2.12 is the next classical result that we will also use several times:

Proposition 2.13 ([13, Lemma 4.2.5]). *Let $\mathcal{F}$ be a C^r ($r \geq 1$) foliation on the annulus $\mathbb{S}^1 \times [0,1]$, transverse to the boundary and without compact leaf. Then there is a smooth surjection $\theta : \mathbb{S}^1 \times [0,1] \to [0,1]$ mapping $\mathbb{S}^1 \times \{0\}$ on 0 and $\mathbb{S}^1 \times \{1\}$ on 1 and so that $\mathcal{F}$ is transverse to the fibers of θ.*

As we did not find a reference for the precise statement of Proposition 2.12, we explain its proof below.

Hint for the proof of Proposition 2.12. One first notices that the surjection θ given by Proposition 2.13 can be chosen so that $\theta(x,t) = t$ for t close to 0 or to 1.

Let us fix such surjections $\theta_{\mathcal{F}}$ and $\theta_{\mathcal{G}}$ associated to $\mathcal{F}$ and $\mathcal{G}$ by Proposition 2.13.

We get a map $\varphi_{\mathcal{F}} : \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1 \times [0,1]$ defined as

$$\varphi_{\mathcal{F}}(x,t) = (y, \theta_{\mathcal{F}}(x,t)),$$

where $(y,0)$ is the intersection of the leaf of $\mathcal{F}$ through (x,t) with $\mathbb{S}^1 \times \{0\}$. We define a map $\varphi_{\mathcal{G}}$ in the same way.

As $\mathcal{F}$ coincides with $\mathcal{G}$ in a neighborhood of the boundary and $\mathcal{F}, \mathcal{G}$ have the same holonomy map, and as $\theta_{\mathcal{F}}$ coincides with $\theta_{\mathcal{G}}$ close to the boundary, one easily checks that $\varphi_{\mathcal{F}}$ coincides with $\varphi_{\mathcal{G}}$ in a neighborhood of the boundary. Now the announced map φ is just

$$\varphi = \varphi_{\mathcal{G}}^{-1} \circ \varphi_{\mathcal{F}}.$$

One easily checks that φ satisfies all the announced properties. □

A classical consequence of Proposition 2.13 is that a foliation on $\mathbb{T}^2$ admitting a complete transversal is conjugated to the suspension of the first return map on this transversal.

3. Existence of a complete transversal for two transverse foliations without parallel compact leaves

In this section, we consider two foliations $\mathcal{F}$, $\mathcal{G}$ on the torus $\mathbb{T}^2$ which do not have parallel compact leaves (see Definition 1.1). According to Proposition 2.5, the foliations $\mathcal{F}$ and $\mathcal{G}$ have no Reeb component. In particular, $\mathcal{F}$ and $\mathcal{G}$ are orientable. By Theorem 2.6, for each of them, there exist complete transverse cross sections. In this section, we prove that any two transverse foliations without parallel compact leaves share a complete transverse cross section.

Proposition 3.1. *If two C^1 foliations $\mathcal{F}$ and $\mathcal{G}$ are transverse on $\mathbb{T}^2$ and have no parallel compact leaves, then there exists a smooth simple closed curve γ which is a complete transversal to both $\mathcal{F}$ and $\mathcal{G}$.*

Proof. As noticed before the statement of Proposition 3.1, the foliations $\mathcal{F}$ and $\mathcal{G}$ have no Reeb component and therefore $\mathcal{F}$ and $\mathcal{G}$ are orientable. Thus there exist two unit vector fields X, Y such that X and Y are tangent to the foliations $\mathcal{F}$ and $\mathcal{G}$ respectively.

Since X and Y are transverse, the vector field $\frac{1}{2}X + \frac{1}{2}Y$ is transverse to both $\mathcal{F}$ and $\mathcal{G}$. Let Z be a smooth vector field C^0 close enough to $\frac{1}{2}X + \frac{1}{2}Y$ so that Z is non-singular and transverse to both foliations $\mathcal{F}$ and $\mathcal{G}$. Furthermore, up to perform a small perturbation, we can assume that Z admits a periodic orbit $\tilde{\gamma}$ which is a simple closed curve transverse to both $\mathcal{F}$ and $\mathcal{G}$.

According to Lemma 2.2, if $\tilde{\gamma}$ is not a complete transversal of one of the foliations $\mathcal{F}$ or $\mathcal{G}$, this foliation admits a compact leaf homotopic to $\tilde{\gamma}$. As $\mathcal{F}$ and $\mathcal{G}$ have no parallel compact leaves, this may happen to at most one of $\mathcal{F}$ and $\mathcal{G}$. In other words, $\tilde{\gamma}$ is a complete transversal for at least one of the foliations, thus we assume that $\tilde{\gamma}$ is a complete transversal for $\mathcal{F}$. If $\tilde{\gamma}$ is a complete transversal for $\mathcal{G}$, we are done.

Thus we assume that it is not the case. Therefore Lemma 2.2 implies that $\mathcal{G}$ has compact leaves which are in the homotopy class of $\tilde{\gamma}$. We denote by $C_{\mathcal{G}}$ a compact leaf of $\mathcal{G}$, and we denote by L a segment of a leaf of $\mathcal{F}$ with endpoints p, q on $\tilde{\gamma}$ and whose interior is disjoint from $\tilde{\gamma}$; furthermore, if $\mathcal{F}$ has a compact leaf, we choose L contained in a compact leaf of $\mathcal{F}$. We denote by $\sigma \subset \tilde{\gamma}$ the (unique) non trivial oriented segment so that

- σ joins the final point q of L to its initial point p;
- the interior of σ is disjoint from $\{p, q\}$;
- the orientation of σ coincides with the transverse orientation of the foliation $\mathcal{G}$, given by the vector field X directing $\mathcal{F}$.

Thus the concatenation $\gamma_0 = L \cdot \sigma$ is a closed curve (which is simple unless $p = q$ in that case $p = q$ is the unique and non topologically transverse intersection point) consisting of one leaf segment and one transverse segment to $\mathcal{F}$. A classical process allows us to smooth γ_0 into a smooth curve γ transverse to $\mathcal{F}$ (see Figure 3 for the case $p \neq q$ and Figure 4 for the case $p = q$), and the choice of the oriented segment σ allows us to choose γ transverse to $\mathcal{G}$. Furthermore, we have that

- γ cuts the compact leaf $C_{\mathcal{G}}$ of $\mathcal{G}$ transversely and in exactly one point;
- if $\mathcal{F}$ has compact leaves, then γ cuts the compact leaf containing L transversely and in exactly 1 point;
- γ is a closed simple curve (even in the case $p = q$).

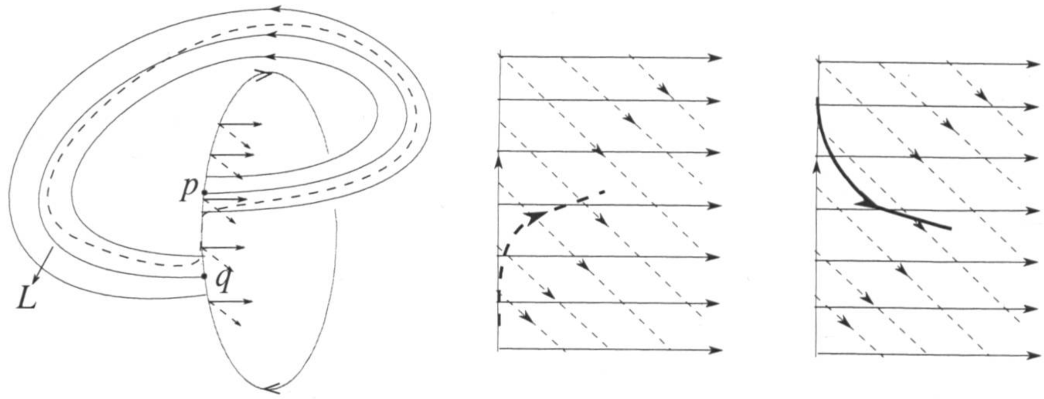

Figure 3. In the first figure: the dash line is the transversal γ; the dash and real arrows on the circle pointing outside give the orientations of $\mathcal{G}$ and $\mathcal{F}$ respectively. The second and the third figure show the good choice of curve and bad choice of curve respectively.

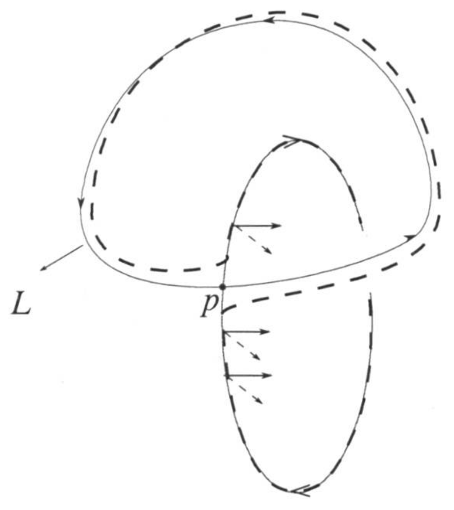

Figure 4. The dash line is the transversal γ. The dash and real arrows on the circle pointing outside give the orientations of $\mathcal{G}$ and $\mathcal{F}$ respectively.

Now γ is a simple closed curve transverse to $\mathcal{G}$ and has non-vanishing intersection number with a compact leaf of $\mathcal{G}$, and therefore γ is not homotopic to the compact leaves of $\mathcal{G}$. Lemma 2.2 implies therefore that γ is a complete transversal of $\mathcal{G}$. The same argument show that, if $\mathcal{F}$ has a compact leaf, then γ is a complete transversal of $\mathcal{F}$. Finally, if $\mathcal{F}$ has no compact leaves, any closed transversal is a complete transversal, ending the proof. □

4. Deformation of a foliation along its transverse foliation

For any C^1 foliation $\mathcal{E}$, we will denote by $\mathcal{E}_x$ the leaf of $\mathcal{E}$ through x. For any two points x, y on a common leaf of $\mathcal{E}$, we denote $d_{\mathcal{E}}(x, y)$ as the distance between x, y on the $\mathcal{E}$-leaf.

Proposition 4.1. *Let $S = \mathbb{R} \times [0, 1]$ be a horizontal strip on $\mathbb{R}^2$. Assume that $\widetilde{\mathcal{E}}$, $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ are C^1 foliations on S satisfying that*

- *the foliation $\widetilde{\mathcal{G}}$ is transverse to $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{E}}$, that is, $\widetilde{\mathcal{E}} \pitchfork \widetilde{\mathcal{G}}$ and $\widetilde{\mathcal{F}} \pitchfork \widetilde{\mathcal{G}}$;*
- *the foliations $\widetilde{\mathcal{E}}$, $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ are invariant under the map $(r, s) \mapsto (r + 1, s)$;*
- *the foliations $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{F}}$ have the same holonomy map from $\mathbb{R} \times \{0\}$ to $\mathbb{R} \times \{1\}$;*
- *Each leaf of each foliation intersects the two boundary components of S transversely.*

Then there exists a continuous family $\{\Phi_t\}_{t\in[0,1]}$ of C^1 diffeomorphisms on $\mathbb{R} \times [0, 1]$ such that

- $\Phi_0 = \mathrm{Id}$;
- $\Phi_1(\widetilde{\mathcal{E}}) = \widetilde{\mathcal{F}}$;
- $\Phi_t(\widetilde{\mathcal{E}}) \pitchfork \widetilde{\mathcal{G}}$, *for every $t \in [0, 1]$;*
- Φ_t *commutes with the map $(r, s) \mapsto (r + 1, s)$, for any $t \in [0, 1]$;*
- Φ_t *coincides with the identity map on $\mathbb{R} \times \{0, 1\}$, for any $t \in [0, 1]$.*

If furthermore $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{F}}$ coincide in a neighborhood of the boundary $\mathbb{R} \times \{0, 1\}$ of S then we can choose the family $\{\Phi_t\}_{t\in[0,1]}$ of diffeomorphisms so that there is a neighborhood of $\mathbb{R} \times \{0, 1\}$ on which the Φ_t coincides with the identity map, for any $t \in [0, 1]$.

Proof. By assumption, for each $x \in \mathbb{R} \times \{0\}$, the leaf $\widetilde{\mathcal{E}}_x$ and the leaf $\widetilde{\mathcal{F}}_x$ have the same boundary.

Claim 4.2. *For each $y \in \widetilde{\mathcal{E}}_x$, the leaf $\widetilde{\mathcal{G}}_y$ intersects $\widetilde{\mathcal{F}}_x$ in a unique point.*

Proof. Since $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{F}}$ are transverse to $\widetilde{\mathcal{G}}$, one can prove that every leaf of $\widetilde{\mathcal{G}}$ intersects every leaf of $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{F}}$ in at most one point. If y is an end point of $\widetilde{\mathcal{E}}_x$, it is also an end point of $\widetilde{\mathcal{F}}_x$, concluding. Consider now y in the interior of $\widetilde{\mathcal{E}}_x$.

Recall that $\widetilde{\mathcal{G}}_y$ is a segment joining the two boundary components of S. Thus $S \setminus \widetilde{\mathcal{G}}_y$ has two connected components. Moreover each connected component of $S \setminus \widetilde{\mathcal{G}}_y$ contains exactly one end point of $\widetilde{\mathcal{E}}_x$. As $\widetilde{\mathcal{F}}_x$ has the same end points as $\widetilde{\mathcal{E}}_x$, it intersects $\widetilde{\mathcal{G}}_y$. $\square$

Now, we can define a map $h_{\widetilde{\mathcal{E}}}$ from S to itself. For each $x \in S$, there exists a unique leaf of $\widetilde{\mathcal{F}}$ which has the same boundary as $\widetilde{\mathcal{E}}_x$, by the claim above, $\widetilde{\mathcal{G}}_x$ intersects that unique leaf of $\widetilde{\mathcal{F}}$ in only one point and we denote it as $h_{\widetilde{\mathcal{E}}}(x)$ (see Figure 5 below).

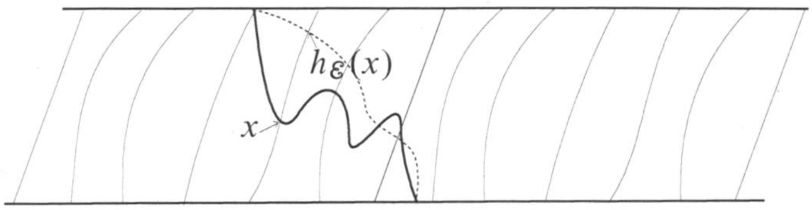

Figure 5. The light line, the dark line and dash line denote the leaves of $\widetilde{\mathcal{G}}$, $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{F}}$ respectively.

Since $\widetilde{\mathcal{E}}$, $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ are C^1-foliations, $h_{\widetilde{\mathcal{E}}}$ is a C^1 map, and its inverse $h_{\widetilde{\mathcal{F}}}$ is obtained by reversing the roles of $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{F}}$, proving that $h_{\widetilde{\mathcal{E}}}$ is a diffeomorphism. Since each foliation is invariant under horizontal translation $(r,s) \mapsto (r+1,s)$, the diffeomorphisms $h_{\widetilde{\mathcal{E}}}$ and $h_{\widetilde{\mathcal{F}}}$ commute with the map $(r,s) \mapsto (r+1,s)$.

Since x and $h_{\widetilde{\mathcal{E}}}(x)$ are on the same $\widetilde{\mathcal{G}}$ leaf, the map $d_{\mathcal{G}}(x, h_{\widetilde{\mathcal{E}}}(x))$ is well defined from S to $\mathbb{R}$ and one can check that it is a C^1 map which is invariant under the translation $(r,s) \mapsto (r+1,s)$.

Now, for each $t \in [0,1]$, we define $\Phi_t(x)$ as the point, in the segment joining x to $h_{\widetilde{\mathcal{E}}}(x)$ in the leaf $\widetilde{\mathcal{G}}_x$, so that

$$d_{\widetilde{\mathcal{G}}}(x, \Phi_t(x)) = t \cdot d_{\widetilde{\mathcal{G}}}(x, h_{\widetilde{\mathcal{E}}}(x))$$

and

$$d_{\widetilde{\mathcal{G}}}(\Phi_t(x), h_{\widetilde{\mathcal{E}}}(x)) = (1-t) \cdot d_{\widetilde{\mathcal{G}}}(x, h_{\widetilde{\mathcal{E}}}(x)).$$

Then, we have that $\Phi_0 = \mathrm{Id}$, $\Phi_1 = h_{\widetilde{\mathcal{E}}}$ and Φ_t commutes with the horizontal translation $(r,s) \mapsto (r+1,s)$ and preserves each leaf of the foliation $\widetilde{\mathcal{G}}$. One easily checks that Φ_t is of class C^1 and depends continuously on t. Furthermore, its derivative along the leaves of $\widetilde{\mathcal{G}}$ does not vanish, so that Φ_t is a diffeomorphism restricted to every leaf of $\widetilde{\mathcal{G}}$. As Φ_t preserves every leaf of $\widetilde{\mathcal{G}}$, one deduces that $\Phi_t(\widetilde{\mathcal{E}})$ is transverse to $\widetilde{\mathcal{G}}$ and Φ_t is a diffeomorphism of S. Thus, $\{\Phi_t\}_{t\in[0,1]}$ is the announced continuous path of C^1 diffeomorphisms of S. $\square$

For any C^1 simple closed curve γ on $\mathbb{T}^2$ whose homotopy class is non-trivial, we can cut the torus along γ to get a cylinder. The universal cover of the cylinder is a strip denoted by S_γ and diffeomorphic to $\mathbb{R} \times [0,1]$. For any C^1 foliation $\mathcal{E}$ on $\mathbb{T}^2$ transverse to γ, one denotes by $\widetilde{\mathcal{E}}$ the lift of $\mathcal{E}$ on S_γ.

Corollary 4.3. *Let $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ be three C^1 foliations on $\mathbb{T}^2$. Assume that:*

- $\mathcal{E} \pitchfork \mathcal{G}$ *and* $\mathcal{F} \pitchfork \mathcal{G}$;
- *there exists a C^1 simple closed curve γ such that*
  (1) *the curve γ is a complete transversal of the foliations $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$;*
  (2) *the lifted foliations $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{F}}$ have the same holonomy map defined from one boundary component of S_γ to the other;*

Then there exists a continuous family of C^1 diffeomorphisms $\{\Phi_t\}_{t\in[0,1]} \subset \mathrm{Diff}^1(\mathbb{T}^2)$ such that

- $\Phi_0 = \mathrm{Id}$;
- $\Phi_t(\mathcal{E}) \pitchfork \mathcal{G}$, *for every $t \in [0,1]$;*
- $\Phi_1(\mathcal{E}) = \mathcal{F}$.

Sketch of proof. If we just apply Proposition 4.1, one obtains a family of homeomorphisms of $\mathbb{T}^2$ which are C^1 diffeomorphisms on the complement of γ and which coincide with the identity map on γ and satisfy all the announced properties. Thus the unique difficulty is the regularity along γ. For that we check that the construction in the proof of Proposition 4.1 can be done on the whole universal cover of $\mathbb{T}^2$ commuting with all the deck transformations, leading to diffeomorphisms on $\mathbb{T}^2$. $\square$

5. Deformation process for transverse foliations without parallel compact leaves: proof of Theorem 1.2

5.1. Separating transverse foliations by two linear ones and proof of Theorem 1.2.

Theorem 5.1. *Let $\mathcal{F}$ and $\mathcal{G}$ be two transverse C^1 foliations on $\mathbb{T}^2$ without parallel compact leaves. Then there are two affine foliations $\mathcal{H}$ and $\mathcal{I}$ on $\mathbb{T}^2$ and a diffeomorphism $\theta : \mathbb{T}^2 \to \mathbb{T}^2$ so that*

- *the foliations $\theta(\mathcal{F})$, $\theta(\mathcal{G})$, $\mathcal{H}$ and $\mathcal{I}$ are pairwise transverse;*
- *there are local orientations of the foliations at any point $p \in \mathbb{T}^2$ so that*
  - *$\theta(\mathcal{F})$ and $\theta(\mathcal{G})$ cut $\mathcal{H}$ with the same orientation;*
  - *$\theta(\mathcal{F})$ and $\theta(\mathcal{G})$ cut $\mathcal{I}$ with opposite orientations.*

The two affine foliations $\mathcal{H}$ and $\mathcal{I}$ divide the tangent space T_pM at each point $p \in \mathbb{T}^2$ into four quadrants, and Theorem 5.1 asserts that the tangent lines at p of $\theta(\mathcal{F})$ and $\theta(\mathcal{G})$ are contained in different quadrants.

The proof of Theorem 5.1 is the aim of the whole section. Let us first deduce the proof of Theorem 1.2

Proof of Theorem 1.2. We consider two transverse C^1 foliations $\mathcal{F}$ and $\mathcal{G}$ on $\mathbb{T}^2$ without parallel compact leaves, the diffeomorphism θ and the affine foliations $\mathcal{H}$ and $\mathcal{I}$ given by Theorem 5.1. Consider any vector $u \in \mathbb{R}^2$ and let T_u be the affine translation of $\mathbb{T}^2$ directed by u, that is $T_u(p) = p + u$.

Claim 5.2. *For any $u \in \mathbb{R}^2$, the foliation $T_u(\theta(\mathcal{F}))$ is transverse to $\theta(\mathcal{G})$*

Proof. The foliations $\mathcal{H}$ and $\mathcal{I}$ are invariant by T_u, and the quadrants defined by $\mathcal{H}$ and $\mathcal{I}$ are preserved by T_u so that $T_u(\theta(\mathcal{F}))$ is still transverse to both $\mathcal{H}$ and $\mathcal{I}$ and its tangent bundle is contained in the same quadrants as $\theta(\mathcal{F})$, and therefore $T_u(\theta(\mathcal{F}))$ is not contained in the same quadrants as the tangent bundle of $\theta(\mathcal{G})$.

Thus $T_u(\theta(\mathcal{F}))$ is transverse to $\theta(\mathcal{G})$, concluding. □

Consider $(m, n) \in \mathbb{Z}^2 = H_1(\mathbb{T}^2, \mathbb{Z})$ and let $u = (r, s)$ be the image of (m, n) by the natural action of θ on $H_1(\mathbb{T}^2, \mathbb{Z})$. Then the announced loop of diffeomorphisms is $\{\varphi_t = \theta^{-1} T_{tu} \theta\}_{t\in[0,1]}$. Then $\varphi_t(\mathcal{F})$ is transverse to $\mathcal{G}$ for every $t \in [0, 1]$, $\varphi_0 = \varphi_1 = \mathrm{id}_{\mathbb{T}^2}$, and the loop $t \mapsto \varphi_t(p)$ is in the homology class of (m, n) for every $p \in \mathbb{T}^2$. □

Therefore, it remains to prove Theorem 5.1. The proof is divided into two main steps corresponding to the next subsections.

5.2. Separating transverse foliations by a circle bundle. In this section, consider two transverse foliations $\mathcal{F}$ and $\mathcal{G}$ without parallel compact leaves. We first choose a coordinate to make $\mathcal{G}$ in a "good position", then we apply Proposition 4.1 to deform $\mathcal{F}$ in "good position", keeping $\mathcal{G}$ invariant.

By Lemma 3.1, there exists a smooth simple closed curve γ which is a complete transversal of $\mathcal{F}$ and $\mathcal{G}$. The aim of this section is to prove next result which can be seen as the first step for proving Theorem 5.1.

Theorem 5.3. *Let $\mathcal{F}$ and $\mathcal{G}$ be two transverse C^1 foliations on $\mathbb{T}^2$ and assume that they share the same complete transversal γ. Then there exists $\theta \in \mathrm{Diff}^1(\mathbb{T}^2)$ such that*

- $\theta(\gamma) = \mathbb{S}^1 \times \{0\}$;
- *Both $\theta(\mathcal{F})$ and $\theta(\mathcal{G})$ are transverse to the horizontal circle $\mathbb{S}^1 \times \{t\}$, for any $t \in \mathbb{S}^1$.*

Up to now, $\mathcal{F}$ and $\mathcal{G}$ are two transverse foliations on $\mathbb{T}^2$ which share the same complete transversal γ. In particular, $\mathcal{G}$ is conjugated to the suspension of its holonomy (first return map) on γ. In other words, we can choose an appropriate coordinate on $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ such that:

- the circle $\gamma = \mathbb{S}^1 \times \{0\}$ is a complete transversal for $\mathcal{F}$ and $\mathcal{G}$;
- the foliation $\mathcal{G}$ is everywhere transverse to the horizontal circles;
- the foliation $\mathcal{G}$ is vertical in a small neighborhood of $\mathbb{S}^1 \times \{0\}$.

Under this coordinate, we cut the torus along γ and we get a cylinder $\mathbb{S}^1 \times [0,1]$. Thus $\mathbb{T}^2$ is obtained from $\mathbb{S}^1 \times [0,1]$ by identifying $(x,0)$ with $(x,1)$, for $x \in \mathbb{S}^1$.

Now, we take a universal cover of that cylinder, we get a strip $S = \mathbb{R} \times [0,1]$. The foliations $\mathcal{F}$ and $\mathcal{G}$ can be lifted as two foliations $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ on S, respectively. Moreover, $\widetilde{\mathcal{G}}$ is everywhere transverse to the horizontal direction.

The proof of Theorem 5.3 has two steps: first we build a foliation $\mathcal{E}$ on $\mathbb{T}^2$ transverse to $\mathcal{G}$ and to the horizontal foliation, so that $\mathcal{E}$ has the same holonomy as $\mathcal{F}$. Then we push $\mathcal{F}$ on $\mathcal{E}$ by a diffeomorphism preserving $\mathcal{G}$, by using Proposition 4.1. Thus the main step of the proof is:

Proposition 5.4. *With the notations above, there exist $\epsilon > 0$ and a C^1 foliation $\mathcal{E}$ transverse to $\mathcal{G}$ on $\mathbb{T}^2$ such that the foliation $\widetilde{\mathcal{E}}$ induced by $\mathcal{E}$ on the strip $S = \mathbb{R}\times[0,1]$ satisfies:*

- *the foliation $\widetilde{\mathcal{E}}$ is transverse to the horizontal direction;*
- *For any $x \in \mathbb{R} \times \{0\}$, we have that*

$$\widetilde{\mathcal{E}}_x \cap \big(\mathbb{R} \times \big([0,\epsilon] \cup [1-\epsilon, 1]\big)\big) = \widetilde{\mathcal{F}}_x \cap \big(\mathbb{R} \times \big([0,\epsilon] \cup [1-\epsilon, 1]\big)\big).$$

Remark 5.5. The last item of Proposition 5.4 means that

- the foliations $\mathcal{E}$ and $\mathcal{F}$ coincide in a neighborhood of γ
- the holonomy maps from $\mathbb{R} \times \{0\}$ to $\mathbb{R} \times \{1\}$ associated to $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{F}}$ are the same.

Proof. We denote by

$$f, g : \mathbb{R} \times \{0\} \mapsto \mathbb{R} \times \{1\}$$

the C^1 holonomy maps of $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ respectively.

As $\widetilde{\mathcal{F}}$ is transverse to $\widetilde{\mathcal{G}}$, we have that $f(x) \neq g(x)$, for any $x \in \mathbb{R}\times\{0\}$. Hence, we can assume that $f(x) > g(x)$ for any $x \in \mathbb{R} \times \{0\}$ (the other case is similar).

We denote by $g_t : \mathbb{R} \to \mathbb{R}$ the holonomy of $\widetilde{\mathcal{G}}$ from $\mathbb{R} \times \{0\}$ to $\mathbb{R} \times \{t\}$. In particular, g_0 is the identity map and $g_1 = g$. Our assumption that $\mathcal{G}$ is vertical close to the boundary, implies that g_t is the identity map for t small enough and $g_t = g$ for t close to 1.

Let $\psi_0 : S \to S$ be defined by $(x,t) \mapsto (g_t(x), t)$. Then ψ_0 is a diffeomorphism which commutes with the translation $T_1 : (x,t) \mapsto (x+1, t)$.

Consider the foliations $\widetilde{\mathcal{G}_0} = \psi_0^{-1}(\tilde{G})$ and $\widetilde{\mathcal{F}_0} = \psi_0^{-1}(\widetilde{\mathcal{F}})$. Now we have:

- $\widetilde{\mathcal{G}_0}$ is the vertical foliation;
- $\widetilde{\mathcal{F}_0}$ is a C^1 foliation transverse to the vertical foliation and transverse to the boundary of S, and invariant by the translation T_1. We denote by $\mathcal{F}_0$ its quotient on the annulus $\mathbb{S}^1 \times [0,1]$.
- every leaf of $\widetilde{\mathcal{F}_0}$ goes from $\mathbb{R}\times\{0\}$ to $\mathbb{R}\times\{1\}$ so that the holonomy map f^0 is well defined and $f^0 = g^{-1} \circ f$. Our assumption $f(x) > g(x)$ means that $f^0(x) > x$ for every x.

As $\mathcal{F}_0$ is transverse to the boundary of $\mathbb{S}^1 \times [0,1]$, there is $\delta > 0$ so that $\widetilde{\mathcal{F}_0}$ is transverse to the horizontal foliation on $\mathbb{R} \times [0,\delta]$ and on $\mathbb{R} \times [1-\delta, 1]$. Thus the holonomy $f_t^0 : \mathbb{R} \times \{0\} \to \mathbb{R} \times \{t\}$ of the foliation $\widetilde{\mathcal{F}_0}$ is well defined for $t \in [0,\delta] \cup [1-\delta,1]$ and satisfies:

- $f_t^0(x) > x$ for $t > 0$, and moreover $f_{t_1}^0(x) < f_{t_2}^0(x)$ for $t_1, t_2 \in [0,\delta] \cup [1-\delta, 1]$ and $t_1 < t_2$.
- The map $(x,t) \mapsto f_t^0(x)$ is C^1 and $\frac{\partial f_t^0(x)}{\partial t} > 0$ (because $\widetilde{\mathcal{F}_0}$ is transverse to the vertical foliation).

Consider $\varepsilon > 0$ so that:

- $$\varepsilon < \inf_{x\in\mathbb{R}} \{f_{1-\delta}^0(x) - f_\delta^0(x), \text{ for } x \in \mathbb{R}\}.$$
- $$\varepsilon < \inf \left\{ \frac{\partial f_t^0(x)}{\partial t}, \text{ for } x \in \mathbb{R} \text{ and } t \in [0,\delta] \cup [1-\delta, 1] \right\}.$$

With this choice of ε, one can easily check the following inequalities.

Claim 5.6.

- *For any $t \in [0,\delta]$ and $x \in \mathbb{R}$, one has*
$$f_t^0(x) < \frac{f_{1-\delta}^0(x) + f_\delta^0(x)}{2} + \left(t - \frac{1}{2}\right)\varepsilon; \tag{5.1}$$
- *For any $t \in [1-\delta, 1]$ and $x \in \mathbb{R}$, one has*
$$\frac{f_{1-\delta}^0(x) + f_\delta^0(x)}{2} + \left(t - \frac{1}{2}\right)\varepsilon < f_t^0(x). \tag{5.2}$$

Let $\alpha : [0,1] \to [0,1]$ be a smooth function so that:

- $\alpha(t) \equiv 1$, for t close to 0 and close to 1;
- $\alpha(t) \equiv 0$, for $t \in [\frac{\delta}{2}, 1 - \frac{\delta}{2}]$;
- $\frac{d\alpha}{dt} \leq 0$ on $[0,\delta]$ and $\frac{d\alpha}{dt} \geq 0$ on $[1-\delta, 1]$.

For $x \in \mathbb{R}$ and $t \in [0,1]$, we define $h_t(x)$ as follows:

- If $t \in [0,\delta] \cup [1-\delta, 1]$, then
$$h_t(x) = \alpha(t) f_t^0(x) + (1-\alpha(t))\left(\frac{f_\delta^0(x) + f_{1-\delta}^0(x)}{2} + \varepsilon\left(t - \frac{1}{2}\right)\right);$$
- if $t \in [\delta, 1-\delta]$, then
$$h_t(x) = \frac{f_\delta^0(x) + f_{1-\delta}^0(x)}{2} + \varepsilon\left(t - \frac{1}{2}\right).$$

Claim 5.7. *The map $\psi_1 : (x,t) \mapsto (h_t(x), t)$ is well defined and is a C^1 diffeomorphism of S such that:*

- *ψ_1 preserves the horizontal foliation and commutes with the translation T_1;*
- $$\frac{\partial h_t(x)}{\partial t} > 0, \text{ for every } (x,t) \in S.$$

Proof. One easily checks that ψ_1 is continuous and of class C^1. The formula gives also that ψ_1 commutes with T_1. Now

$$\frac{\partial h_t(x)}{\partial x} = \frac{1}{2}\left(\frac{\partial f_\delta^0(x)}{\partial x} + \frac{\partial f_{1-\delta}^0(x)}{\partial x}\right) > 0 \ \text{ if } t \in [\delta, 1-\delta]$$

and

$$\begin{aligned}\frac{\partial h_t(x)}{\partial x} &= \alpha(t)\frac{\partial f_t^0(x)}{\partial x} + (1-\alpha(t)) \cdot \frac{1}{2}\left(\frac{\partial f_\delta^0(x)}{\partial x} + \frac{\partial f_{1-\delta}^0(x)}{\partial x}\right) \\ &> 0 \ \text{ if } t \notin [\delta, 1-\delta].\end{aligned}$$

This shows that h_t is a diffeomorphism of $\mathbb{R}$ for every $t \in [0,1]$.

It remains to prove the last item of the claim. One can observe that:

$$\frac{\partial h_t(x)}{\partial t} = \varepsilon > 0, \ \text{ if } t \in [\delta, 1-\delta]$$

and if $t \notin [\delta, 1-\delta]$ the derivative $\frac{\partial h_t(x)}{\partial t}$ is equal to:

$$\frac{d\alpha(t)}{dt}\left(f_t^0(x) - \frac{1}{2}\left(f_\delta^0(x) + f_{1-\delta}^0(x) + \varepsilon\left(t - \frac{1}{2}\right)\right)\right) + \alpha(t)\frac{\partial f_t^0}{\partial t} + (1-\alpha(t))\varepsilon.$$

The last two terms of this sum are positive, as product of positive numbers.

For $t \in [0, \delta]$, the first term is product of two negative numbers, as the derivative of α is negative and (5.1) implies:

$$f_t^0(x) - \frac{1}{2}\left(f_\delta^0(x) + f_{1-\delta}^0(x) + \varepsilon\left(t - \frac{1}{2}\right)\right) < 0.$$

For $t \in [1-\delta, 1]$, the first term is product of two positive numbers, as the derivative of α is positive and (5.2) implies:

$$f_t^0(x) - \frac{1}{2}\left(f_\delta^0(x) + f_{1-\delta}^0(x) + \varepsilon\left(t - \frac{1}{2}\right)\right) > 0.$$

Thus $\frac{\partial h_t(x)}{\partial t} > 0$ for every (x,t). □

Now the foliation $\widetilde{\mathcal{H}}$ defined as the image of the vertical foliation by ψ_1 satisfies:

- $\widetilde{\mathcal{H}}$ is transverse to the horizontal foliation.

- $\widetilde{\mathcal{H}}$ is transverse to the vertical foliation (that is, to $\widetilde{\mathcal{G}}_0$).
- its holonomy from $\mathbb{R} \times \{0\}$ to $\mathbb{R} \times \{t\}$ is h_t. In particular, it coincides with f_t for t so that $\alpha(t) = 1$, that is, in the neighborhood of $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$.
- as a consequence of the previous item, the foliation $\widetilde{\mathcal{H}}$ coincides with $\widetilde{\mathcal{F}}_0$ in the neighborhood of $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$.

We can now finish the proof of Proposition 5.4: the announced foliation on the strip S is $\widetilde{\mathcal{E}} = \psi_0(\widetilde{\mathcal{H}})$. This foliation is invariant under the translation T_1, so it passes to the quotient in a foliation $\mathcal{E}$ on the annulus $\mathbb{S}^1 \times [0,1]$. As $\widetilde{\mathcal{E}}$ coincides with $\widetilde{\mathcal{F}}$ in a neighborhood of the boundary of S, one gets that $\mathcal{E}$ coincides with $\mathcal{F}$ on the boundary of the annulus, and therefore this foliation induces a C^1 foliation, still denoted by $\mathcal{E}$ on the torus $\mathbb{T}^2$. □

Next remark ends the proof of Theorem 5.3:

Remark 5.8. According to Proposition 4.1, there is a continuous path of diffeomorphisms $\{\varphi_s\}_{s\in[0,1]}$ of S so that:

- φ_s commutes with the translation $T_1 : (x,t) \mapsto (x+1,t)$;
- φ_0 is the identity map;
- for every $s \in [0,1]$, the diffeomorphism φ_s coincides with the identity map in a neighborhood of the boundary of S;
- $\varphi_s(\widetilde{\mathcal{G}}) = \widetilde{\mathcal{G}}$ for every s; in particular $\varphi_s(\widetilde{\mathcal{F}})$ is transverse to $\widetilde{\mathcal{G}}$ for every s;
- $\varphi_1(\widetilde{\mathcal{F}}) = \widetilde{\mathcal{E}}$.

We now state a small variation of the statement of Theorem 5.3 which follows (exactly as Theorem 5.3) from of Propositions 5.4 and 4.1, and that we will use in a next section.

Lemma 5.9. *Let $\mathcal{F}$ and $\mathcal{G}$ be two transverse C^1-foliations on the annulus $\mathbb{S}^1 \times [0,1]$. Assume that*

- *$\mathcal{G}$ is transverse to every circle $\mathbb{S}^1 \times \{t\}$;*
- *$\mathcal{F}$ is transverse to the boundary $\mathbb{S}^1 \times \{0,1\}$ and has no compact leaf in $\mathbb{S}^1 \times (0,1)$.*

Then there is a C^1 diffeomorphism θ of $\mathbb{S}^1 \times [0,1]$ which coincides with the identity map in a neighborhood of the boundary and which preserves every leaf of $\mathcal{G}$, so that $\theta(\mathcal{F})$ is transverse to every circle $\mathbb{S}^1 \times \{t\}$.

5.3. Building the second linear foliation: end of the proof of Theorem 5.1.

Proposition 5.10. *Let $\mathcal{F}$ and $\mathcal{G}$ be two transverse C^1 foliations on $\mathbb{T}^2$ without parallel compact leaves. Assume that both $\mathcal{F}$ and $\mathcal{G}$ are transverse to the horizontal foliation. We endow $\mathcal{F}$ and $\mathcal{G}$ with orientations so that they cut the horizontal foliation with the same orientation.*

Then there exists a smooth (C^∞) foliation $\mathcal{E}$ on $\mathbb{T}^2$ such that:

- *the circle $\mathbb{S}^1 \times \{0\}$ is a complete transversal to $\mathcal{E}$;*
- *the holonomy map induced by $\mathcal{E}$ on γ has a diophantine rotation number;*
- *the foliation $\mathcal{E}$ is transverse to the foliations $\mathcal{F}$, $\mathcal{G}$ and to the horizontal direction. We endow it with an orientation so that it cuts the horizontal foliation with the same orientation as $\mathcal{F}$ and $\mathcal{G}$.*
- *the foliation $\mathcal{E}$ cuts $\mathcal{F}$ and $\mathcal{G}$ with opposite orientations.*

Proof. As already done before, we cut the torus along $\mathbb{S}^1 \times \{0\}$, getting an annulus, and we denote by $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ the lift of $\mathcal{F}$ and $\mathcal{G}$ on the strip $\mathbb{R} \times [0,1]$ which is the universal cover of the annulus. We denote by f and g the holonomy maps from $\mathbb{R} \times \{0\}$ to $\mathbb{R} \times \{1\}$ associated to the lifted foliations $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$, respectively. By transversality of $\mathcal{F}$ with $\mathcal{G}$, we have that either $f(x) < g(x)$ for any x, or $f(x) > g(x)$ for any x. Without loss of generality, we assume that $f(x) < g(x)$. Let $\tau(f)$ and $\tau(g)$ be the translation numbers of f and g.

Claim 5.11. $\tau(f) \neq \tau(g)$.

Proof. We prove it by contradiction. Assume that $\tau(f) = \tau(g)$, then $\tau(f) = \tau(g)$ is either rational or irrational. When they are irrational, since $f(x) < g(x)$, by Proposition 2.10, we have that $\tau(f) < \tau(g)$, a contradiction. When they are both rational, then there exist $m, n \in \mathbb{N}$ such that $\tau(f) = \tau(g) = \frac{n}{m}$. Hence, there exist two points $x_0, y_0 \in \mathbb{R}$, such that

$$f^m(x_0) = x_0 + n \text{ and } g^m(y_0) = y_0 + n,$$

which implies that there exist compact leaves of $\mathcal{F}$ and $\mathcal{G}$ that are in the homotopy class of (m, n), contradicting the non-parallel assumption. □

We endow $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ with orientations such that they point inward the strip S at $\mathbb{R} \times \{0\}$ and point outward at $\mathbb{R} \times \{1\}$, and $\mathcal{F}$ and $\mathcal{G}$ are endowed with the corresponding orientations. Let X and Y be the unit vector fields tangent to $\mathcal{F}$ and $\mathcal{G}$ respectively, pointing to the orientation of the corresponding foliation, and $\widetilde{X}$ and $\widetilde{Y}$ be their lifts on S.

Claim 5.12. *There are smooth vector fields U and V on $\mathbb{T}^2$ so that*

- *at each point $x \in \mathbb{T}^2$, the vertical coordinates of $U(x)$ and $V(x)$ are strictly positive. In particular, U and V are transverse to the horizontal foliation. We denote by $\widetilde{U}$ and $\widetilde{V}$ the lifts of U and V on the strip S.*
- *let h and k be the holonomies of $\widetilde{U}$ and $\widetilde{V}$, respectively, from $\mathbb{R} \times \{0\}$ to $\mathbb{R} \times \{1\}$. These holonomies commute with the translation T_1, and let $\tau(h)$ and $\tau(k)$ denote their translation numbers. Then*

$$\tau(f) \leq \tau(h) < \tau(k) \leq \tau(g).$$

Proof. Just consider a small enough $\varepsilon > 0$ and consider smooth vector fields U and V arbitrarily C^0 close to $X + \varepsilon Y$ and $\varepsilon X + Y$, respectively. $\square$

Now the vector fields $U_t = (1-t)U + tV$, $t \in [0,1]$, are all transverse to both foliations $\mathcal{F}$, $\mathcal{G}$ and to the horizontal foliation, and they cut $\mathcal{F}$ and $\mathcal{G}$ with opposite orientations. We denote by $\widetilde{U}_t$ the lift of U_t on the strip S. Let τ_t denote the translation number of the holonomy of $\widetilde{U}_t$ from $\mathbb{R} \times \{0\}$ to $\mathbb{R} \times \{1\}$. According to Proposition 2.9, the map $t \mapsto \tau_t$ is a continuous monotonous function joining $\tau(h)$ to $\tau(k)$. As $\tau(h) < \tau(k)$, there is $t \in (0,1)$ for which τ_t is an irrational diophantine number, ending the proof. $\square$

We end the proof of Theorem 5.1 by noticing that Theorem 2.11 implies

Lemma 5.13. *Let $\mathcal{E}$ be a smooth foliation on $\mathbb{T}^2$ transverse to the horizontal foliation and so that its holonomy on $\mathbb{S}^1 \times \{0\}$ is a diffeomorphism with an irrational diophantine rotation number. Then there is a diffeomorphism θ of $\mathbb{T}^2$ which preserves each horizontal circle $\mathbb{S}^1 \times \{t\}$, for any $t \in \mathbb{S}^1$, and satisfies that $\theta(\mathcal{E})$ is an affine foliation.*

6. Deformation process of parallel case and proof of Theorem 1.3

We dedicate this whole section to give the proof of Theorem 1.3. We state a definition which is only used in this section.

Definition 6.1. Given a C^1 foliation $\mathcal{E}$ on the annulus $[0,1] \times \mathbb{S}^1 = [0,1] \times \mathbb{R}/\mathbb{Z}$ without compact leaves such that $\mathcal{E}$ is transverse to the vertical circle $\{t\} \times \mathbb{S}^1$, for any $t \in [0,1]$. The leaves of such a foliation $\mathcal{E}$ are called

- *not increasing (resp. not decreasing)*, if the lifted foliation $\widetilde{\mathcal{E}}$ on $[0,1] \times \mathbb{R}$ satisfies that every leaf of $\widetilde{\mathcal{E}}$ is not increasing (resp. not decreasing);
- *non-degenerate increasing (resp. non-degenerate decreasing)*, if the lifted foliation $\widetilde{\mathcal{E}}$ on $[0,1] \times \mathbb{R}$ satisfies that every leaf of $\widetilde{\mathcal{E}}$ is strictly increasing (resp. strictly decreasing) and transverse to the horizontal foliation $\{[0,1] \times \{t\}\}_{t \in \mathbb{R}}$.

6.1. Normal form for two transverse foliations with parallel compact leaves and proof of Theorem 1.3. The aim of this section is the proof of Theorem 1.3. The main step for this proof is the following result which puts any pair of transverse C^1 foliations in a canonical position.

Theorem 6.2. *Let $\mathcal{F}$ and $\mathcal{G}$ be two transverse C^1 foliations on $\mathbb{T}^2$ admitting parallel compact leaves. Then there are an integer k, a set of points $\{t_i\}_{i \in \mathbb{Z}/k\mathbb{Z}}$ in $\mathbb{S}^1$ which are cyclically ordered on $\mathbb{S}^1$, and a diffeomorphism $\theta : \mathbb{T}^2 \to \mathbb{T}^2$ so that*

- *the foliations $\theta(\mathcal{F})$ and $\theta(\mathcal{G})$ are transverse to $\{t_i\} \times S^1$, for any $i \in \mathbb{Z}/k\mathbb{Z}$;*

- *for each* $i \in \mathbb{Z}/k\mathbb{Z}$*, the restrictions of the foliations* $\theta(\mathcal{F})$ *and* $\theta(\mathcal{G})$ *to the annulus* $C_i = [t_i, t_{i+1}] \times \mathbb{S}^1$ *satisfy one of the six possibilities below*
  (1) $\theta(\mathcal{F})$ *coincides with the horizontal foliation on* C_i *and* $\theta(\mathcal{G})$ *admits compact leaves in* C_i*;*
  (2) $\theta(\mathcal{G})$ *coincides with the horizontal foliation on* C_i *and* $\theta(\mathcal{F})$ *admits compact leaves in* C_i*;*
  (3) *the foliations* $\theta(\mathcal{F})$ *and* $\theta(\mathcal{G})$ *are transverse to the vertical foliation on* C_i*. Furthermore, every leaf of* $\theta(\mathcal{F})$ *(resp. of* $\theta(\mathcal{G})$*) on* C_i *is non-degenerate increasing (resp. not increasing);*
  (4) *the foliations* $\theta(\mathcal{F})$ *and* $\theta(\mathcal{G})$ *are transverse to the vertical foliation on* C_i*. Furthermore, every leaf of* $\theta(\mathcal{F})$ *(resp. of* $\theta(\mathcal{G})$*) on* C_i *is not increasing (resp. non-degenerate increasing);*
  (5) *the foliations* $\theta(\mathcal{F})$ *and* $\theta(\mathcal{G})$ *are transverse to the vertical foliation on* C_i*. Furthermore, every leaf of* $\theta(\mathcal{F})$ *(resp. of* $\theta(\mathcal{G})$*) on* C_i *is non-degenerate decreasing (resp. not decreasing);*
  (6) *the foliations* $\theta(\mathcal{F})$ *and* $\theta(\mathcal{G})$ *are transverse to the vertical foliation on* C_i*. Furthermore, every leaf of* $\theta(\mathcal{F})$ *(resp. of* $\theta(\mathcal{G})$*) on* C_i *is not decreasing (resp. non-degenerate decreasing).*

The proof of Theorem 6.2 will be done in the next subsections. We start below by ending the proof of Theorem 1.3.

Proof of Theorem 1.3. Let $\mathcal{F}$ and $\mathcal{G}$ be two C^1 foliations of $\mathbb{T}^2$ admitting parallel compact leaves, and let $\alpha \in \pi_1(\mathbb{T}^2)$ be the homotopy class of the compact leaves of $\mathcal{F}$ and $\mathcal{G}$. Let $k > 0$, $\{t_i\}_{i\in\mathbb{Z}/k\mathbb{Z}}$ and θ be the integer, the elements of $\mathbb{S}^1$ and the diffeomorphism given by Theorem 6.2, respectively.

One easily checks that there is at least one annulus of the type (1) or (2). As a consequence, the compact leaves of $\theta(\mathcal{F})$ are isotopic to the vertical circle $\{0\} \times \mathbb{S}^1$.

Consider any vertical vector $(0, t)$, for $t \in \mathbb{R}$, and let V_t be the vertical translation defined by $(r, s) \mapsto (r, t + s)$. Then V_t preserves each annulus C_i. Now one can check, on each annulus C_i, that $V_t(\theta(\mathcal{F}))$ is transverse to $\theta(\mathcal{G})$.

Consider now $\beta \in \langle\alpha\rangle$, so that $\beta = n\alpha$ for some $n \in \mathbb{Z}$. Then the announced loop of diffeomorphisms is $\{\theta^{-1} \circ V_{nt} \circ \theta\}_{t\in[0,1]}$. □

6.2. First decomposition in annuli. By Theorem 2.7, the sets of compact leaves of $\mathcal{F}$ and $\mathcal{G}$ are all compact sets. We denote the unions of compact leaves of $\mathcal{F}$ and $\mathcal{G}$ as K_F and K_G respectively. Note that every compact leaf of $\mathcal{F}$ is disjoint from any compact leaf of $\mathcal{G}$, because they are in the same homotopy class, and by assumption, $\mathcal{F}$ and $\mathcal{G}$ are transverse. Thus K_G and K_F are disjoint compact sets.

The aim of this section is to prove Proposition 6.3 below which is an important step for proving Theorem 6.2.

Proposition 6.3. *Let $\mathcal{F}$ and $\mathcal{G}$ be two transverse C^1-foliations on $\mathbb{T}^2$ having parallel compact leaves. Then there are k_0 and a family $\{B_i\}_{i\in\mathbb{Z}/4k_0\mathbb{Z}}$ of annuli so that*

- *each B_i is an annulus diffeomorphic to $[0,1]\times\mathbb{S}^1$ and embedded in $\mathbb{T}^2$ whose boundary is transverse to both foliations $\mathcal{F}$ and $\mathcal{G}$.*
- *B_i is disjoint from B_j if $j\notin\{i-1,i,i+1\}$, and $B_i\cap B_{i+1}$ consists in a common connected component of the boundaries ∂B_i and ∂B_{i+1}. In particular, the interiors of these B_i are pairwise disjoint;*
- *each annulus B_{2j+1} is disjoint from the compact leaves of $\mathcal{F}$ and of $\mathcal{G}$, that is*

$$B_{2j+1}\cap(K_F\cup K_G)=\emptyset;$$

- *each annulus B_{4i} contains compact leaves of $\mathcal{F}$ and is disjoint from the compact leaves of $\mathcal{G}$;*
- *each annulus B_{4i+2} contains compact leaves of $\mathcal{G}$ and is disjoint from the compact leaves of $\mathcal{F}$.*

We say that a compact set C is *a $\mathcal{F}$-annulus* (resp. *a $\mathcal{G}$-annulus*) if we have the following:

– the compact set C is diffeomorphic to either $\mathbb{S}^1$ or $\mathbb{S}^1\times[0,1]$;

– the compact set C is disjoint from K_G (resp. of K_F);

– the boundary of C consists of compact leaves of $\mathcal{F}$ (resp. of $\mathcal{G}$).

We say that two compact leaves L_1,L_2 of $\mathcal{F}$ (resp. of $\mathcal{G}$) are *K_G-homotopic* (resp. *K_F-homotopic*) if $L_1\cup L_2$ bounds a $\mathcal{F}$-annulus (resp. *a $\mathcal{G}$-annulus*).

Remark 6.4. • The union of two non-disjoint $\mathcal{F}$-annuli is a $\mathcal{F}$-annulus.

- two compact leaves of $\mathcal{F}$ are K_G-homotopic if and only if they are contained in the same $\mathcal{F}$-annulus.
- there is $\delta>0$ so that any two compact leaves of $\mathcal{F}$ passing through points x,y with $d(x,y)\le\delta$ are K_G-homotopic.

As a direct consequence of Remark 6.4, one gets

Lemma 6.5. (1) *The relation of K_G-homotopy (resp. of K_F-homotopy) is an equivalence relation on K_F (resp. of K_G).*

(2) *there are finitely many K_G-homotopy classes (resp. K_F-homotopy classes).*

(3) *There are $k\in\mathbb{N}\setminus\{0\}$ and pairwise disjoint compact sets $\{A_i\}_{i\in\mathbb{Z}/2k\mathbb{Z}}$, so that*

- *A_{2i} is a $\mathcal{F}$-annulus and A_{2i+1} is a $\mathcal{G}$-annulus.*
- *For each K_G-homotopy class (resp. K_F-homotopy class) of compact leaves of $\mathcal{F}$ (resp. of $\mathcal{G}$), there is a (unique) i so that the class is precisely the set of compact leaves of $\mathcal{F}$ (resp. of $\mathcal{G}$) contained in A_{2i} (resp. in A_{2i+1}).*

- *these $\{A_i\}$ are cyclically ordered in the following meaning: for any $i \in \mathbb{Z}/2k\mathbb{Z}$, the set $\mathbb{T}^2 \setminus (A_{i-1} \cup A_{i+1})$ consists precisely of two disjoint open annuli such that one of them contains A_i and is disjoint from A_j for $j \neq i$.*

Proof. The set $\{A_i\}$ is defined as the set of the unions of all the $\mathcal{F}$-annuli containing compact leaves of $\mathcal{F}$ in a given K_G homotopy class and the unions of all the $\mathcal{G}$-annuli containing compact leaves of $\mathcal{G}$ in a given K_F homotopy class. Then $\{A_i\}$ bound a family of disjoint compact annuli (or circles) whose boundary are non-null homotopic simple curves on $\mathbb{T}^2$. Thus these curves are in the same homotopy class and the annuli are cyclically ordered on $\mathbb{T}^2$. Thus, up to reorder the annuli, we assume that the order is compatible with the cyclic order. Finally if A_i is a $\mathcal{F}$ annulus then A_{i+1} cannot be a $\mathcal{F}$-annulus, otherwise there would exist a $\mathcal{F}$-annulus containing both A_i and A_{i+1}, contradicting to the maximality of A_i. □

The annuli A_{2i} and A_{2i+1} will be called the *maximal $\mathcal{F}$-annuli* and *maximal $\mathcal{G}$-annuli*, respectively.

Lemma 6.6. *With the hypotheses and terminology above, each maximal $\mathcal{F}$-annulus (resp. maximal $\mathcal{G}$-annulus) A admits a base of neighborhoods $\{B_n\}_{n\in\mathbb{N}}$ which are diffeomorphic to $[0,1] \times \mathbb{S}^1$ and whose boundaries are transverse to both $\mathcal{F}$ and $\mathcal{G}$.*

Proof. Assume for instance that A is a maximal $\mathcal{F}$-annulus. Its boundary consists of compact leaves of $\mathcal{F}$, and in particular is transverse to $\mathcal{G}$. Furthermore any neighborhood V of A contains an annulus U which is a neighborhood of A and satisfies that $U \setminus A$ is disjoint from K_F and K_G. Now each connected component of $U \setminus A$ contains an embedded circle which consists in exactly one segment of leaf of $\mathcal{F}$ and one segment of leaf of $\mathcal{G}$. Exactly as in Section 3, we get a simple closed curve transverse to both $\mathcal{F}$ and $\mathcal{G}$ by smoothing such a curve.

One gets the announced annulus by considering such a transverse curve to both $\mathcal{F}$ and $\mathcal{G}$ in each connected component of $U \setminus A$. □

Proof of Proposition 6.3. The announced annuli B_{4i} and B_{4i+2} are pairwise disjoint neighborhoods of the maximal $\mathcal{F}$-annuli and maximal $\mathcal{G}$-annuli, respectively, given by Lemma 6.6. Each annulus B_{2j+1} is given by the closure of a connected component of $\mathbb{T}^2 \setminus \bigcup_i (B_{4i} \cup B_{4i+2})$. □

6.3. In the neighborhoods of the maximal $\mathcal{F}$ annuli. The aim of this section is to prove the following Proposition which implies that, in the neighborhoods of the maximal $\mathcal{F}$ and $\mathcal{G}$-annuli, one can put $\mathcal{F}$ and $\mathcal{G}$ in the position announced by Theorem 6.2.

Proposition 6.7. *Let $\mathcal{F}$ and $\mathcal{G}$ be two transverse foliations on $\mathbb{T}^2$ having parallel compact leaves. Let $\{B_j\}_{\in\mathbb{Z}/4k\mathbb{Z}}$ be the annuli, which are built in Proposition 6.3 and whose boundaries are transverse to both $\mathcal{F}$ and $\mathcal{G}$, and A_{4i} (resp. A_{4i+2}) be the maximal $\mathcal{F}$-annuli (resp. $\mathcal{G}$-annuli) contained in B_{4i} (resp. in B_{4i+2}), for $i \in \mathbb{Z}/k\mathbb{Z}$.*

Then there exists $\theta \in \mathrm{Diff}^1(\mathbb{T}^2)$ *such that for every* $j \in \mathbb{Z}/4k\mathbb{Z}$, *one has*

$$\theta(B_j) = \left[\frac{j}{4k}, \frac{j+1}{4k}\right] \times \mathbb{S}^1;$$

and for every $i \in \mathbb{Z}/k\mathbb{Z}$, *one has:*

- *the foliation* $\theta(\mathcal{G})$ *coincides with the horizontal foliation* $\{[\frac{4i}{4k}, \frac{4i+1}{4k}] \times \{t\}\}_{t\in\mathbb{S}^1}$ *on* $\theta(B_{4i}) = [\frac{4i}{4k}, \frac{4i+1}{4k}] \times \mathbb{S}^1$;
- *there are* $\frac{4i}{4k} < a_{4i} \le b_{4i} < \frac{4i+1}{4k}$ *such that* $\theta(A_{4i}) = [a_{4i}, b_{4i}] \times \mathbb{S}^1$. *In particular,* $\{a_{4i}\} \times \mathbb{S}^1$ *and* $\{b_{4i}\} \times \mathbb{S}^1$ *are compact leaves of* $\theta(\mathcal{F})$;
- *the foliation* $\theta(\mathcal{F})$ *is transverse to the vertical circle* $\{r\} \times \mathbb{S}^1$, *for any* $r \in [\frac{4i}{4k}, a_{4i}) \cup (b_{4i}, \frac{4i+1}{4k}]$.

and similarly:

- *the foliation* $\theta(\mathcal{F})$ *coincides with the horizontal foliation* $\{[\frac{4i+2}{4k}, \frac{4i+3}{4k}] \times \{t\}\}_{t\in\mathbb{S}^1}$ *on* $\theta(B_{4i+2}) = [\frac{4i+2}{4k}, \frac{4i+3}{4k}] \times \mathbb{S}^1$.
- *there are* $\frac{4i+2}{4k} < a_{4i+2} \le b_{4i+2} < \frac{4i+3}{4k}$ *so that* $\theta(A_{4i+2}) = [a_{4i+2}, b_{4i+2}] \times \mathbb{S}^1$. *In particular,* $\{a_{4i+2}\} \times \mathbb{S}^1$ *and* $\{b_{4i+2}\} \times \mathbb{S}^1$ *are compact leaves of* $\theta(\mathcal{G})$;
- *the foliation* $\theta(\mathcal{G})$ *is transverse to the vertical circle* $\{r\} \times \mathbb{S}^1$, *for any* $r \in [\frac{4i+2}{4k}, a_{4i+2}) \cup (b_{4i+2}, \frac{4i+3}{4k}]$.

Proposition 6.7 is a straightforward consequence of Lemma 6.8 below:

Lemma 6.8. *Let* $\mathcal{F}$ *and* $\mathcal{G}$ *be two transverse* C^1*-foliations of the annulus* $[0,1] \times \mathbb{S}^1$ *so that the boundary* $\{0,1\} \times \mathbb{S}^1$ *is transverse to both* $\mathcal{F}$ *and* $\mathcal{G}$. *Assume that* $\mathcal{G}$ *has no compact leaves (in* $(0,1) \times \mathbb{S}^1$*) and* $\mathcal{F}$ *admits compact leaves in* $(0,1) \times \mathbb{S}^1$.

Then there exists $\theta \in \mathrm{Diff}^1([0,1] \times \mathbb{S}^1)$ *so that*

(1) *the foliation* $\theta(\mathcal{G})$ *is the horizontal foliation* $\{[0,1] \times \{t\}\}_{t\in\mathbb{S}^1}$.

(2) *there are* $0 < a \le b < 1$ *so that* $\{a\} \times \mathbb{S}^1$ *and* $\{b\} \times \mathbb{S}^1$ *are compact leaves of* $\theta(\mathcal{F})$ *and every compact leaf of* $\theta(\mathcal{F})$ *is contained in* $[a,b] \times \mathbb{S}^1$;

(3) *the foliation* $\theta(\mathcal{F})$ *is transverse to the vertical circle* $\{r\} \times \mathbb{S}^1$ *for* $r \notin [a,b]$.

The proof of Lemma 6.8 uses the Lemma 6.9 below.

Lemma 6.9. *For any continuous function* $\varphi : [0,1] \mapsto [0,+\infty)$ *such that* $\varphi > 0$ *on* $(0,1)$, *and any interval* $(c,d) \subset (0,1) \subset \mathbb{S}^1$, *there exists* $\theta \in \mathrm{Diff}^\infty(\mathbb{R} \times \mathbb{S}^1)$ *such that*

- *the diffeomorphism* θ *coincides with the identity map out of* $[0,1] \times \mathbb{S}^1$;
- $\theta([0,1] \times \{y\}) = [0,1] \times \{y\}$;
- $D\theta(\frac{\partial}{\partial y}) = \frac{\partial}{\partial y} + a(x,y)\frac{\partial}{\partial x}$;
- $a(x,y) > 0$, *for any* $(x,y) \in (0,1) \times (c,d)$;
- $a(x,y) > -\varphi(x)$, *for any* $(x,y) \in (0,1) \times \mathbb{S}^1$.

Proof. We fix $c < d$ and take a point $e \in (d, 1)$. We take

$$\theta(x, y) = (x + \alpha(x)\beta(y), y)$$

where $\alpha : \mathbb{R} \to [0, +\infty)$ and $\beta : \mathbb{S}^1 \to [0, 1]$ are smooth functions so that

- $\alpha(x)$ is defined on $\mathbb{R}$ and equals to zero in $(-\infty, 0] \cup [1, +\infty)$;
- $0 < \alpha(x) < \varphi(x)$ in the set $(0, 1)$ and $\alpha \equiv 0$ out of $[0, 1]$ (the existence of such a function is not hard to check);
- the derivative $\alpha'(x)$ is everywhere strictly larger than -1;
- $\beta(y)$ is equal to zero in the set $[0, c] \cup [e, 1]$;
- the derivative $\beta'(y)$ is strictly positive for $y \in (c, d)$;
- the derivative $\beta'(y)$ is larger than -1 everywhere.

With this choice, one gets that the restriction of θ to any horizontal line has a non-vanishing derivative, hence is a diffeomorphism. One deduces that θ is a diffeomorphism of $[0, 1] \times \mathbb{S}^1$. Furthermore, the function $a(x, y)$ in the statement is $\alpha(x)\cdot\beta'(y)$ which is strictly positive on $(0, 1)\times(c, d)$ and larger than $-\alpha(x) > -\varphi(x)$ for $x \in (0, 1)$, concluding the proof. □

Remark 6.10. In the proof of Lemma 6.9, if we define θ_t by

$$\theta_t(x, y) = (x + t\alpha(x)\beta(y), y), \quad \text{for any } t \in [0, 1],$$

one gets a continuous family of diffeomorphisms for the C^∞ topology so that θ_0 is the identity map and every θ_t, $t \neq 0$, satisfies the conclusion of Lemma 6.9. In particular, in Lemma 6.9 one may choose θ arbitrarily C^∞ close to identity.

Proof of Lemma 6.8. As $\mathcal{G}$ is transverse to the boundary and has no compact leaves in $[0, 1]\times\mathbb{S}^1$, then as a simple corollary of Proposition 2.12 one gets that, up to consider the images of $\mathcal{F}$ and $\mathcal{G}$ by a diffeomorphism of the annulus, we may assume that $\mathcal{G}$ is the horizontal foliation and that there are compact leaves $\{a\} \times \mathbb{S}^1$ and $\{b\} \times \mathbb{S}^1$, $0 < a \leq b < 1$, so that the compact leaves of $\mathcal{F}$ are contained in $[a, b] \times \mathbb{S}^1$. In other words, we may assume that items (1) and (2) are already satisfied. It remains to get item (3), that is, to get the transversality of $\mathcal{F}$ with the vertical fibers out of $[a, b] \times \mathbb{S}^1$.

We first show:

Claim 6.11. *There is a C^1 foliation $\mathcal{H}$ defined in a neighborhood of the compact leaf $\{a\} \times \mathbb{S}^1$ so that*

- *the leaves of $\mathcal{H}$ are transverse to the horizontal foliation $\mathcal{G}$;*
- *$\{a\} \times \mathbb{S}^1$ is a compact leaf of $\mathcal{H}$;*

- *the holonomies h and f on the transversal* $[0,1]\times\{0\}$ *for the foliations $\mathcal{H}$ and $\mathcal{F}$ are equal;*
- *the foliation $\mathcal{H}$ is transverse to the vertical circle $\{r\}\times\mathbb{S}^1$, for $r<a$.*

Proof. We fix $0<\epsilon<1/2$ and a function $\alpha:[0,1]\to[0,1]$ so that $\alpha\equiv 0$ in $[0,\varepsilon]$, $\alpha\equiv 1$ in $[1-\varepsilon,1]$ and $\alpha'(s)>0$ for $s\in(\varepsilon,1-\varepsilon)$.

Consider the foliation $\mathcal{H}_0$, defined in a neighborhood of the compact leaf $\{a\}\times\mathbb{S}^1$, whose holonomy map $h_s:[0,1]\times\{0\}\to[0,1]\times\{s\}$, for any $s\in\mathbb{S}^1$, is defined by $r\mapsto\alpha(s)f(r)+(1-\alpha(s))r$, where $f:[0,1]\times\{0\}\to[0,1]\times\{0\}$ is the holonomy map of $\mathcal{F}$.

As $\mathcal{F}$ has no compact leaves on $[0,a)\times\mathbb{S}^1$, one gets that $f(r)\neq r$ for every $r<a$. Thus, by the choice of $\mathcal{H}_0$, we have that:

- $\mathcal{H}_0$ is transverse to the horizontal foliation everywhere;
- $\mathcal{H}_0$ is transverse to the vertical foliation at each point (r,s) with $r<a$ and $s\in(\varepsilon,1-\varepsilon)$;
- $\mathcal{H}_0$ is vertical for s in the interval $[0,\varepsilon]\cup[1-\varepsilon,1]=[-\varepsilon,\varepsilon]\subset\mathbb{R}/\mathbb{Z}=\mathbb{S}^1$.

We fix an interval $[e,f]\subset\mathbb{S}^1$ disjoint from $[-\varepsilon,\varepsilon]$. The foliation $\mathcal{H}_0$ is directed by vectors of the form $\frac{\partial}{\partial s}+\delta(r,s)\frac{\partial}{\partial r}$, where the function δ is continuous and non-vanishing on $[0,a)\times[e,f]$. We define $\varphi(r)=\inf_{s\in[e,f]}|\delta(r,s)|$. By the absolute continuity of δ, the map φ is continuous and positive for $r<a$. The map φ is only defined on a small neighborhood of a, and we extend it to $[0,a]$ as a continuous function which is positive on $(0,a)$.

Applying Lemma 6.9 to φ and to an interval (c,d) containing $[-\varepsilon,\varepsilon]$ and disjoint from $[e,f]$, one gets a smooth diffeomorphism θ_0 of $[0,a]\times\mathbb{S}^1$, preserving each horizontal leaf, such that $\theta_0(\mathcal{H}_0)$ is transverse to the vertical foliation on $[0,a)\times\mathbb{S}^1$, concluding the proof of the claim. □

The foliation $\mathcal{H}$ defined by the claim in a neighborhood of $\{a\}\times\mathbb{S}^1$, is conjugated to $\mathcal{F}$ by a diffeomorphism preserving the compact leaf $\{a\}\times\mathbb{S}^1$ and every horizontal segment $[0,1]\times\{s\}$. We can do the same in a neighborhood of the compact leaf $\{b\}\times\mathbb{S}^1$.

Thus there is a diffeomorphism θ_1 of $[0,1]\times\mathbb{S}^1$ preserving the leaves $\{a\}\times\mathbb{S}^1$ and $\{b\}\times\mathbb{S}^1$ and preserving every horizontal segment $[0,1]\times\{s\}$, and there is $\varepsilon>0$ so that $\theta_1(\mathcal{F})$ is transverse to the vertical circles on $[a-\varepsilon,a)\times\mathbb{S}^1$ and on $(b,b+\varepsilon]\times\mathbb{S}^1$.

For concluding the proof, it remains to put the foliation $\mathcal{F}$ transverse to the vertical circles on the annuli $[0,a-\varepsilon]\times\mathbb{S}^1$ and $[b+\varepsilon,1]\times\mathbb{S}^1$. On each of these annuli, we have that

- $\mathcal{G}$ is a foliation transverse to the circle bundle;
- $\mathcal{F}$ is transverse everywhere to $\mathcal{G}$;
- Both $\mathcal{F}$ and $\mathcal{G}$ are transverse to the boundary and have no compact leaf.

Thus applying Lemma 5.9 to these annuli, one gets diffeomorphisms which preserve each leaf of $\mathcal{G}$ and equal to the identity map on the boundary, such that these diffeomorphisms send $\mathcal{F}$ on a foliation transverse to the circle bundle, concluding the proof. □

Let us add a statement that we will not use, but it is obtained by a slight modification of the proof of Lemma 6.8:

Corollary 6.12. *Let $\mathcal{F}$ and $\mathcal{G}$ be two transverse C^1-foliations of the annulus $[0,1]\times\mathbb{S}^1$ which are both transverse to the boundary. We assume that $\mathcal{G}$ has no compact leaves in the interior of the annulus. Then there is a diffeomorphism θ of the annulus so that $\theta(\mathcal{G})$ is the horizontal foliation $\{[0,1]\times\{s\}\}_{s\in\mathbb{S}^1}$ and $\theta(\mathcal{F})$ satisfies the following properties:*

- *every compact leaf of $\theta(\mathcal{F})$ is a vertical circle;*
- *every non compact leaf of $\theta(\mathcal{F})$ is transverse to the vertical circles.*

Furthermore, θ has the same regularity as $\mathcal{F}$ and $\mathcal{G}$. Finally, if $\mathcal{G}$ is already the horizontal foliation, then θ can be chosen preserving every leaf of $\mathcal{G}$ and equal to the identity map in a neighborhood of the boundary of the annulus.

Proof. The unique change is that, in the last part of the proof, we will need to use Lemma 6.9 in any connected component of the complement of the compact leaves of $\mathcal{F}$, that is, countably many times. For that we uses Remark 6.10 for choosing these diffeomorphisms arbitrarily C^∞-close to identity. □

6.4. Between two maximal $\mathcal{F}$ and $\mathcal{G}$-annuli. The aim of this section is to end the proof of Theorem 6.2 and therefore to end the proof of Theorem 1.3. We consider two transverse C^1 foliations $\mathcal{F}, \mathcal{G}$ on $\mathbb{T}^2$ with parallel compact leaves.

According to Propositions 6.3 and 6.7, there is a diffeomorphism θ_0 of the torus $\mathbb{T}^2$ so that, up to replace $\mathcal{F}$ and $\mathcal{G}$ by $\theta_0(\mathcal{F})$ and $\theta_0(\mathcal{G})$, there is an integer $k > 0$ for which $\mathcal{F}$ and $\mathcal{G}$ satisfy the following properties

- both foliations $\mathcal{F}$ and $\mathcal{G}$ are transverse to every vertical circle $\{\frac{j}{4k}\}\times\mathbb{S}^1$, for any $j\in\mathbb{Z}/4k\mathbb{Z}$;
- both foliations $\mathcal{F}$ and $\mathcal{G}$ have no compact leaves on the vertical annuli $\left[\frac{2i+1}{4k},\frac{2i+2}{4k}\right]\times\mathbb{S}^1$, for any $i\in\mathbb{Z}/2k\mathbb{Z}$;
- the foliation $\mathcal{G}$ coincides with the horizontal foliation on each vertical annulus $\left[\frac{4i}{4k},\frac{4i+1}{4k}\right]\times\mathbb{S}^1$, for any $i\in\mathbb{Z}/k\mathbb{Z}$;
- there are $\frac{4i}{4k} < a_{4i} \le b_{4i} < \frac{4i+1}{4k}$ such that
  - $\{a_{4i}\}\times\mathbb{S}^1$ and $\{b_{4i}\}\times\mathbb{S}^1$ are compact leaves of $\mathcal{F}$;
  - every compact leaf of $\mathcal{F}$ in $\left[\frac{4i}{4k},\frac{4i+1}{4k}\right]\times\mathbb{S}^1$ is contained in $[a_{4i},b_{4i}]\times\mathbb{S}^1$;
  - $\mathcal{F}$ is transverse to the vertical circles on $\left(\left[\frac{4i}{4k},a_{4i}\right)\cup\left(b_{4i},\frac{4i+1}{4k}\right]\right)\times\mathbb{S}^1$;

- the foliation $\mathcal{F}$ coincides with the horizontal foliation on each vertical annulus $\left[\frac{4i+2}{4k}, \frac{4i+3}{4k}\right] \times \mathbb{S}^1$, for any $i \in \mathbb{Z}/k\mathbb{Z}$;
- there are $\frac{4i+2}{4k} < a_{4i+2} \leq b_{4i+2} < \frac{4i+3}{4k}$ such that
  - $\{a_{4i+2}\} \times \mathbb{S}^1$ and $\{b_{4i+2}\} \times \mathbb{S}^1$ are compact leaves of $\mathcal{G}$;
  - every compact leaf of $\mathcal{G}$ in $\left[\frac{4i+2}{4k}, \frac{4i+3}{4k}\right] \times \mathbb{S}^1$ is contained in $[a_{4i+2}, b_{4i+2}] \times \mathbb{S}^1$;
  - $\mathcal{G}$ is transverse to the vertical circles on $\left([\frac{4i+2}{4k}, a_{4i+2}) \cup (b_{4i+2}, \frac{4i+3}{4k}]\right) \times \mathbb{S}^1$.

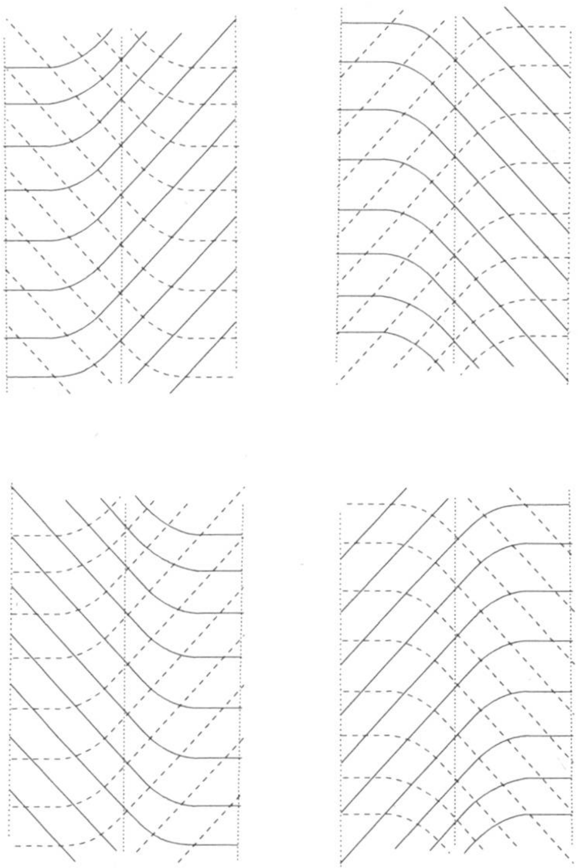

Figure 6. In each figure, the real lines denote the leaves of $\mathcal{F}_i$ and the dash lines denote the leaves of $\mathcal{G}_i$.

The following Proposition ends the proof of Theorem 6.2:

Proposition 6.13. *With the hypotheses and notations above, for any $i \in \mathbb{Z}/2k\mathbb{Z}$, there is a diffeomorphism θ_i of $\mathbb{T}^2$ supported on $(b_{2i}, a_{2i+2}) \times \mathbb{S}^1$ such that for the restrictions $\mathcal{F}_i$ of $\theta_i(\mathcal{F})$ and $\mathcal{G}_i$ of $\theta_i(\mathcal{G})$ to $[b_{2i}, a_{2i+2}] \times \mathbb{S}^1$, we have the followings:*

- *the leaves of both $\mathcal{F}_i$ and $\mathcal{G}_i$ are transverse to every vertical circle $\{r\} \times \mathbb{S}^1$, for any $r \in [b_{2i}, a_{2i+2}]$;*

- *the leaves of $\mathcal{F}_i$ and $\mathcal{G}_i$ satisfy one of the four possibilities below:*
  (1) *the leaves of $\mathcal{F}_i$ (resp. of $\mathcal{G}_i$) are not decreasing (resp. non-degenerate decreasing) on $\left[b_{2i}, \frac{b_{2i}+a_{2i+2}}{2}\right] \times \mathbb{S}^1$ and are non-degenerate increasing (resp. not increasing) on $\left[\frac{b_{2i}+a_{2i+2}}{2}, a_{2i+2}\right] \times \mathbb{S}^1$;*
  (2) *the leaves of $\mathcal{F}_i$ (resp. of $\mathcal{G}_i$) are not increasing (resp. non-degenerate increasing) on $\left[b_{2i}, \frac{b_{2i}+a_{2i+2}}{2}\right] \times \mathbb{S}^1$ and are non-degenerate decreasing (resp. not decreasing) on $\left[\frac{b_{2i}+a_{2i+2}}{2}, a_{2i+2}\right] \times \mathbb{S}^1$;*
  (3) *the leaves of $\mathcal{G}_i$ (resp. of $\mathcal{F}_i$) are not decreasing (resp. non-degenerate decreasing) on $\left[b_{2i}, \frac{b_{2i}+a_{2i+2}}{2}\right] \times \mathbb{S}^1$ and are non-degenerate increasing (resp. not increasing) on $\left[\frac{b_{2i}+a_{2i+2}}{2}, a_{2i+2}\right] \times \mathbb{S}^1$;*
  (4) *the leaves of $\mathcal{G}_i$ (resp. of $\mathcal{F}_i$) are not increasing (resp. non-degenerate increasing) on $[b_{2i}, \frac{b_{2i}+a_{2i+2}}{2}] \times \mathbb{S}^1$ and are non-degenerate decreasing (resp. not decreasing) on $\left[\frac{b_{2i}+a_{2i+2}}{2}, a_{2i+2}\right] \times \mathbb{S}^1$.*

We start by using Proposition 6.13 to end the proof of Theorem 6.2.

Proof of Theorem 6.2. Let $\{\theta_\ell\}_{\ell\in\mathbb{Z}/2k\mathbb{Z}}$ be the sequence of diffeomorphisms on annuli, which are given by Proposition 6.13. We take four sets of points $\{d_{4i}\}_{i\in\mathbb{Z}/k\mathbb{Z}}$, $\{c_{4i+2}\}_{i\in\mathbb{Z}/k\mathbb{Z}}$, $\{d_{4i+2}\}_{i\in\mathbb{Z}/k\mathbb{Z}}$ and $\{c_{4i+4}\}_{i\in\mathbb{Z}/k\mathbb{Z}}$ on $\mathbb{S}^1$ such that

- $$b_{4i} < d_{4i} < \frac{b_{4i}+a_{4i+2}}{2} < c_{4i+2} < a_{4i+2};$$
- $$b_{4i+2} < d_{4i+2} < \frac{b_{4i+2}+a_{4(i+1)}}{2} < c_{4(i+1)} < a_{4(i+1)};$$
- The set $\{c_{4i+2}, c_{4i+4}, d_{4i+2}, d_{4i+2}\}_{i\in\mathbb{Z}/k\mathbb{Z}}$ is disjoint from the union of the supports of all $\{\theta_\ell\}_{\ell\in\mathbb{Z}/2k\mathbb{Z}}$.

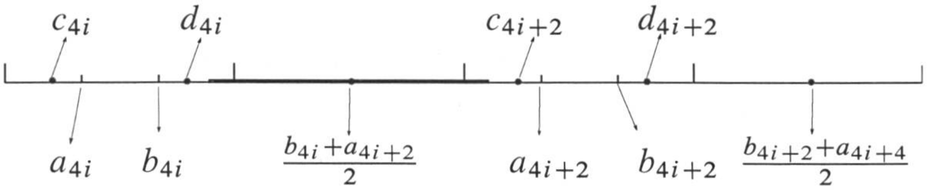

Figure 7. The thick segment denotes the support of some θ_ℓ.

We choose the annuli $\{C_j\}_{j\in\mathbb{Z}/6k\mathbb{Z}}$ as follows

- each annulus C_{6i} is the vertical annulus $[c_{4i}, d_{4i}] \times \mathbb{S}^1$; notice that it contains $[a_{4i}, b_{4i}] \times \mathbb{S}^1$ in its interior;
- each annulus C_{6i+1} is the vertical annulus $[d_{4i}, \frac{1}{2}(b_{4i} + a_{4i+2})] \times \mathbb{S}^1$;
- each annulus C_{6i+2} is the vertical annulus $[\frac{1}{2}(b_{4i} + a_{4i+2}), c_{4i+2}] \times \mathbb{S}^1$;

- each annulus C_{6i+3} is the vertical annulus $[c_{4i+2}, d_{4i+2}] \times \mathbb{S}^1$ containing $[a_{4i+2}, b_{4i+2}] \times \mathbb{S}^1$ in its interior;
- each annulus C_{6i+4} is the vertical annulus $[d_{4i+2}, \frac{1}{2}(b_{4i+2} + a_{4(i+1)})] \times \mathbb{S}^1$;
- each annulus C_{6i+5} is the vertical annulus $[\frac{1}{2}(b_{4i+2} + a_{4(i+1)}), c_{4(i+1)}] \times \mathbb{S}^1$. □

It remains to prove Proposition 6.13.

Lemma 6.14. *Let $\mathcal{F}$ and $\mathcal{G}$ be two transverse foliations on $[0,1] \times \mathbb{S}^1$ so that:*

- *$\{0\} \times \mathbb{S}^1$ is a compact leaf of $\mathcal{F}$;*
- *$\{1\} \times \mathbb{S}^1$ is a compact leaf of $\mathcal{G}$;*
- *$\mathcal{F}$ and $\mathcal{G}$ have no compact leaves in $(0,1) \times \mathbb{S}^1$;*
- *there is a neighborhood $U_0 = [0, \varepsilon_0] \times \mathbb{S}^1$ of $\{0\} \times \mathbb{S}^1$ on which $\mathcal{G}$ coincides with the horizontal foliation and $\mathcal{F}$ is transverse to the vertical circles;*
- *there is a neighborhood $U_1 = [1 - \varepsilon_0, 1]$ of $\{1\} \times \mathbb{S}^1$ on which $\mathcal{F}$ coincides with the horizontal foliation and $\mathcal{G}$ is transverse to the vertical circles.*

Then for any $0 < \varepsilon < \varepsilon_0$ the holonomies of $\mathcal{F}$ and $\mathcal{G}$ from $\Sigma_{0,\varepsilon} = \{\varepsilon\} \times \mathbb{S}^1$ to $\Sigma_{1,\varepsilon} = \{1 - \varepsilon\} \times \mathbb{S}^1$ are well defined. Consider the lifts $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ of $\mathcal{F}$ and $\mathcal{G}$ on the universal cover $[0,1] \times \mathbb{R}$. The holonomies f_ε and g_ε of $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ from $\{\varepsilon\} \times \mathbb{R}$ to $\{1 - \varepsilon\} \times \mathbb{R}$ are well defined. Then for any $\varepsilon > 0$ small enough one has:

$$(f_\varepsilon(x) - x) \cdot (g_\varepsilon(x) - x) < 0, \quad \textit{for every } x \in \mathbb{R}.$$

Proof. On $U_0 \setminus \{0\} \times \mathbb{S}^1$, the foliation $\mathcal{F}$ is transverse to the horizontal segments and to the vertical circles. Therefore its leaves are either non-degenerate increasing or non-degenerate decreasing curves. Let us assume that they are non-degenerate increasing (the other case is similar).

Notice that, on $[\varepsilon_0, 1 - \varepsilon_0] \times \mathbb{S}^1$, the foliations $\mathcal{F}$ and $\mathcal{G}$ are transverse to the boundary and are transverse to each other. We orient $\mathcal{F}$ and $\mathcal{G}$ from $\{\varepsilon_0\} \times \mathbb{S}^1$ to $\{1-\varepsilon_0\} \times \mathbb{S}^1$. As $\mathcal{F}$ is increasing along $\{\varepsilon_0\} \times \mathbb{S}^1$ and horizontal along $\{1-\varepsilon_0\} \times \mathbb{S}^1$, and as $\mathcal{G}$ is horizontal along $\{\varepsilon_0\} \times \mathbb{S}^1$, one gets that $\mathcal{G}$ is decreasing along $\{1-\varepsilon_0\} \times \mathbb{S}^1$. Thus the leaves of $\mathcal{G}$ are decreasing curves on $U_1 \setminus \{1\} \times \mathbb{S}^1$.

Let us denote by

$$\begin{aligned} f_{\varepsilon,0} &: \{\varepsilon\} \times \mathbb{R} \to \{\varepsilon_0\} \times \mathbb{R}, \\ f_{1,\varepsilon} &: \{1 - \varepsilon_0\} \times \mathbb{R} \to \{1 - \varepsilon\} \times \mathbb{R}, \end{aligned}$$

and

$$\begin{aligned} g_{\varepsilon,0} &: \{\varepsilon\} \times \mathbb{R} \to \{\varepsilon_0\} \times \mathbb{R}, \\ g_{1,\varepsilon} &: \{1 - \varepsilon_0\} \times \mathbb{R} \to \{1 - \varepsilon\} \times \mathbb{R} \end{aligned}$$

the holonomies of $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ on the corresponding transversals. We consider them as diffeomorphisms of $\mathbb{R}$ (that is we forget the horizontal coordinate).

Then $g_{\varepsilon,0} = f_{1,\varepsilon}$ are equal to the identity map as they are horizontal foliations in the corresponding regions.

Thus one gets that

$$f_\varepsilon = f_{\varepsilon_0} \circ f_{\varepsilon,0} \text{ and } g_\varepsilon = g_{1,\varepsilon} \circ g_{\varepsilon_0}.$$

Now Lemma 6.14 follows directly from the following claim:

Claim 6.15. *$f_{\varepsilon,0}(x) - x$ and $g_{1,\varepsilon}(x) - x$ converge uniformly to $+\infty$ and $-\infty$, respectively, as ε tends to 0.*

The claim follows directly from the fact that the leaves of $\widetilde{\mathcal{F}}$ (resp. $\widetilde{\mathcal{G}}$) are non-degenerate increasing (resp. non-degenerate decreasing) curves asymptotic to the vertical line $\{0\}\times\mathbb{R}$ (resp. $\{1\}\times\mathbb{R}$) according to the negative orientation (resp. positive orientation). □

One ends the proof of Proposition 6.13 by proving:

Lemma 6.16. *Let $\mathcal{F}$ and $\mathcal{G}$ be two transverse C^1 foliations on the annulus $[0,1]\times\mathbb{S}^1$ which are transverse to the boundary and do not have any compact leaf in the interior. We denote by $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ the lifts of $\mathcal{F}$ and $\mathcal{G}$ to $[0,1]\times\mathbb{R}$. Under that hypotheses, the holonomies of $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ from $\{0\}\times\mathbb{R}$ to $\{1\}\times\mathbb{R}$ are well defined and we denote them as f and g, respectively (and we consider them as diffeomorphisms of $\mathbb{R}$). Assume that:*

- *the foliation $\mathcal{G}$ (resp. $\mathcal{F}$) coincides with the horizontal foliation on a neighborhood of $\{0\}\times\mathbb{S}^1$ (resp. $\{1\}\times\mathbb{S}^1$);*
- *for every $x\in\mathbb{R}$, one has $f(x) > x$ and $g(x) < x$.*

Then there is a diffeomorphism θ of $[0,1]\times\mathbb{S}^1$, equal to the identity map on a neighborhood of the boundary, and isotopic to the identity relative to the boundary, and so that (denoting by $\widetilde{\mathcal{F}}_\theta$ and $\widetilde{\mathcal{G}}_\theta$ the lifts of $\theta(\mathcal{F})$ and $\theta(\mathcal{G})$ to $[0,1]\times\mathbb{R}$):

- *the leaves of $\theta(\mathcal{F})$ and of $\theta(\mathcal{G})$ are transverse to the vertical circles;*
- *the leaves $\widetilde{\mathcal{F}}_\theta$ are non-degenerate increasing on $[0,\frac{1}{2}]\times\mathbb{R}$ and are not decreasing on $[\frac{1}{2},1]\times\mathbb{R}$;*
- *the leaves of $\widetilde{\mathcal{G}}_\theta$ are not increasing on $[0,\frac{1}{2}]\times\mathbb{R}$ and are non-degenerate decreasing on $[\frac{1}{2},1]\times\mathbb{R}$.*

Sketch of proof. We just need to choose a pair of transverse C^1 foliations $\mathcal{F}_0$ and $\mathcal{G}_0$ so that, denoting by $\widetilde{\mathcal{F}}_0$ and $\widetilde{\mathcal{G}}_0$ their lifts on $[0,1]\times\mathbb{R}$, one has:

- $\mathcal{F}_0$ and $\mathcal{G}_0$ are transverse to the vertical foliation;
- $\mathcal{F}_0$ and $\mathcal{G}_0$ coincide with $\mathcal{F}$ and $\mathcal{G}$ in a neighborhood of the boundary;
- the holonomies of $\widetilde{\mathcal{F}}_0$ and $\widetilde{\mathcal{G}}_0$ from $\{0\}\times\mathbb{R}$ to $\{1\}\times\mathbb{R}$ are f and g, respectively;

- the leaves $\widetilde{\mathcal{F}_0}$ are non-degenerate increasing on $[0, \frac{1}{2}] \times \mathbb{R}$ and are not decreasing on $[\frac{1}{2}, 1] \times \mathbb{R}$;
- the leaves of $\widetilde{\mathcal{G}_0}$ are not increasing on $[0, \frac{1}{2}] \times \mathbb{R}$ and are non-degenerate decreasing on $[\frac{1}{2}, 1] \times \mathbb{R}$.

The fact that we can choose such a pair of foliations is similar to the proof of Proposition 5.4.

Then the pair $(\mathcal{F}, \mathcal{G})$ is conjugated to $(\mathcal{F}_0, \mathcal{G}_0)$ by a unique diffeomorphism equal to the identity map in a neighborhood of the boundary. The lift $\widetilde{\theta}$ on $[0, 1] \times \mathbb{R}$ of the announced diffeomorphism θ is built as follows: consider a point $p \in [0, 1] \times \mathbb{R}$ and let $q_F(p)$ and $q_G(p)$ be the intersections with $\{0\} \times \mathbb{R}$ of the leaves $\widetilde{\mathcal{F}_p}$ and $\widetilde{\mathcal{G}_p}$ through p. The transversality of $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ implies that $q_F(p)$ is below $q_G(p)$ and $f(q_F(p))$ is over $g(q_G(p))$. As $\widetilde{\mathcal{F}_0}$ and $\widetilde{\mathcal{G}_0}$ have the same holonomies as $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$, one gets that the leaves of $\widetilde{\mathcal{F}_0}$ and of $\widetilde{\mathcal{G}_0}$ through $q_F(p)$ and $q_G(p)$ have a unique intersection point that we denote by $\widetilde{\theta}(p)$. □

7. Dehn twists, transverse foliations and partially hyperbolic diffeomorphisms

The aim of this section is to give the proof of Proposition 1.8 and of Theorem B.

7.1. Transverse foliations on 3-manifolds and the proof of Proposition 1.8. Let M be a closed 3-manifold and $\mathcal{F}$ and $\mathcal{G}$ be transverse codimension one foliations of class C^1 on M. Thus $\mathcal{F}$ and $\mathcal{G}$ intersect each other along a C^1 foliation $\mathcal{E}$ of dimension 1. We assume that there is a torus T embedded in M such that $\mathcal{E}$ is transverse to T, and we denote by $\mathcal{F}_T$ and $\mathcal{G}_T$ the 1-dimensional C^1 foliations on T obtained by intersecting T with $\mathcal{F}$ and $\mathcal{G}$, respectively.

There is a collar neighborhood U of T and an orientation preserving diffeomorphism $\theta : U \to T \times [0, 1]$ inducing the identity map from T to $T \times \{0\}$, so that $\theta(\mathcal{E})$ is the trivial foliation $\{\{p\} \times [0, 1]\}_{p \in T}$. Then $\theta(\mathcal{F})$ and $\theta(\mathcal{G})$ are the product foliations of $\mathcal{F}_T \times [0, 1]$ and $\mathcal{G}_T \times [0, 1]$, respectively (meaning that their leaves are the product by $[0, 1]$ of the leaves of $\mathcal{F}_T$ and $\mathcal{G}_T$, respectively).

Let u be an element of $G_{\mathcal{F}_T, \mathcal{G}_T} \subset \pi_1(T)$. By definition of $G_{\mathcal{F}_T, \mathcal{G}_T}$, there is a loop $\{\varphi_t\}_{t \in [0,1]}$ of C^1 diffeomorphisms of T so that $\varphi_0 = \varphi_1$ is the identity map, $\varphi_t(\mathcal{F}_T)$ is transverse to $\mathcal{G}_T$ and for any $p \in T$, the loop $\{\varphi_t(p)\}_{t \in [0,1]}$ belongs to the homotopy class of u.

We consider the diffeomorphism Φ on $T \times [0, 1]$ defined by $(p, t) \mapsto (\varphi_{\alpha(t)}(p), t)$, where $\alpha : [0, 1] \to [0, 1]$ is a smooth function equal to 0 in a neighborhood of 0 and to 1 in a neighborhood of 1. Then Φ is a Dehn twist directed by u and Φ is the identity map in a neighborhood of the boundary of $T \times [0, 1]$.

Consider $\Phi(\mathcal{F}_T \times [0,1])$. It is a foliation transverse to any torus $T \times \{t\}$ and it induces $\varphi_{\alpha(t)}(\mathcal{F}_T)$ on $T \times \{t\}$. Therefore it is transverse to $\mathcal{G}_T$.

This proves that $\Phi(\mathcal{F}_T \times [0,1])$ is transverse to the foliation $\mathcal{G}_T \times [0,1]$.

Now the announced Dehn twist on M directed by u is the diffeomorphism ψ with support in U and whose restriction to U is $\theta^{-1} \circ \Phi \circ \theta$. By construction, $\psi(\mathcal{F})$ is transverse to $\mathcal{G}$, ending the proof.

7.2. Anosov flows, Dehn twists and partially hyperbolic diffeomorphisms. Let X be a non-transitive Anosov vector field of class at least C^2 on a closed 3-manifold M and we denote by X_t the flow generated by X. According to Proposition 1.9, any family of transverse tori on which X has no return, are contained in a regular level of a smooth Lyapunov function.

Let $L(x) : M \mapsto \mathbb{R}$ be a smooth Lyapunov function of the flow X_t, and let c be a regular value of L. Thus each connected component of $L^{-1}(c)$ is a torus transverse to X.

Let $T_1, \dots, T_k$ be the disjoint transverse tori such that

$$\cup_{i=1}^{k} T_i = L^{-1}(c).$$

Consider the set $M^r = L^{-1}(c, +\infty)$ and $M^a = L^{-1}(-\infty, c)$. Then M^r and M^a are two disjoint open subsets of M and share the same boundary $\cup_{i=1}^{k} T_i$. Since $L(x)$ is strictly decreasing along the positive orbits of the points in the wandering domain, one gets that M^a and M^r are attracting and repelling regions of the vector field X. We denote by $\mathcal{A}$ and $\mathcal{R}$, respectively, the maximal invariant sets of X in M^a and M^r. Thus $\mathcal{A}$ is a hyperbolic (not necessarily transitive) attractor and $\mathcal{R}$ is a hyperbolic (not necessarily transitive) repeller for X.

By [15, Corollary 4], the center stable foliation $\mathcal{F}_X^{cs}$ and center unstable foliation $\mathcal{F}_X^{cu}$ of the Anosov flow X_t are C^1 foliations. For each $i = 1, \dots, k$, we denote by $\mathcal{F}_i^s$ and $\mathcal{F}_i^u$ the C^1 foliation induced by $\mathcal{F}_X^{cs}$ and $\mathcal{F}_X^{cu}$ on T_i respectively.

As X has no return on $\bigcup_i T_i$, the sets $\{X_t(T_1)\}_{t\in\mathbb{R}}, \dots, \{X_t(T_k)\}_{t\in\mathbb{R}}$ are pairwise disjoint embeddings of $T_i \times \mathbb{R}$ into M. As a consequence, for any integer N, the sets $\{X_t(T_1)\}_{t\in[0,N]}, \dots, \{X_t(T_k)\}_{t\in[0,N]}$ are pairwise disjoint and diffeomorphic to $\mathbb{T}^2 \times [0,N]$.

For each i, we define the diffeomorphism

$$\psi_{i,N} : \{X_t(T_i)\}_{t\in[0,N]} \mapsto T_i \times [0,1]$$

by $(X_t(p)) \mapsto (p, t/N)$, for any $p \in T_i$ and $t \in [0,N]$. Thus $D\psi_{i,N}(X) = \frac{1}{N}\frac{\partial}{\partial s}$ is tangent to the vertical segment $\{p\} \times [0,1]$, for any $p \in T_i$.

We fix a smooth function $\alpha(s) : [0,1] \mapsto [0,1]$ such that $\alpha(s)$ is a non-decreasing function on $[0,1]$, equals to 0 in a small neighborhood of 0 and equals to 1 in a small neighborhood of 1. For each i, the group $G_i = G_{\mathcal{F}_i^s, \mathcal{F}_i^u}$ is the subgroup of $\pi_1(\mathbb{T}^2)$ associated to the pair of transverse foliations $(\mathcal{F}_i^s, \mathcal{F}_i^u)$ by Definition 1.5.

Given an element $u_i \in G_i$, let $\{\varphi_t\}_{t\in[0,1]}$ be the loop in $\mathrm{Diff}^1(T_i)$ associated to u_i by Theorem A.

Consider the map $\Phi_i : T_i \times [0,1] \mapsto T_i \times [0,1]$ defined as

$$(x,s) \mapsto (\varphi_{\alpha(s)}(x), s).$$

Hence, the map $\Psi_{i,N} = \psi_{i,N}^{-1} \circ \Phi_i \circ \psi_{i,N}$ is a Dehn twist directed by u_i. Notice that $\Psi_{i,N}$ can be C^1-smoothly extended on the whole manifold M to be the identity map outside $X_t(T_i)$.

The main part of Theorem B is directly implied by the following theorem:

Theorem 7.1. *With the notations above, when N is chosen large enough, the diffeomorphism $\Psi_{k,N} \circ \cdots \circ \Psi_{1,N} \circ X_N$ is absolute partially hyperbolic.*

Proof. We denote

$$\Psi_N = \Psi_{k,N} \circ \cdots \circ \Psi_{1,N}.$$

Then $\Psi_N \circ X_N$ coincides with X_N on the attracting region M^a. Thus $\mathcal{A}$ is the maximal invariant set of $\Psi_N \circ X_N$ in M^a and is an absolute partially hyperbolic attractor. Furthermore the center stable bundle and the strong stable bundle on $\mathcal{A}$ admit unique continuous and $(\Psi_N \circ X_N)$-invariant extensions $E^{cs}_{\mathcal{A}}$ and $E^s_{\mathcal{A}}$, respectively, on M^a which coincide with the tangent bundles of the center stable and strong stable foliations $\mathcal{F}^{cs}_X$ and $\mathcal{F}^s_X$ of the vector field X.

In the same way, $(\Psi_N \circ X_N)^{-1}$ coincides with X_{-N} on the repelling region M^r. Thus $\mathcal{R}$ is still an absolute partially hyperbolic repeller of $\Psi_N \circ X_N$ and its center unstable and strong unstable bundles admit unique continuous and $(\Psi_N \circ X_N)$-invariant extensions $E^{cu}_{\mathcal{R}}$ and $E^u_{\mathcal{R}}$ on M^r which coincide with, respectively, the tangent bundles of $\mathcal{F}^{cu}_X$ and $\mathcal{F}^u_X$.

Notice that the center unstable and strong unstable bundles $E^{cu}_{\mathcal{R}}$ and $E^u_{\mathcal{R}}$, of the repeller $\mathcal{R}$ for $\Psi_N \circ X_N$ extend in a unique way on $M \setminus \mathcal{A}$, just by pushing by the dynamics of $\Psi_N \circ X_N$.

Thus the bundles $E^{cu}_{\mathcal{R}}$, $E^u_{\mathcal{R}}$, $E^{cs}_{\mathcal{A}}$ and $E^s_{\mathcal{A}}$ coincide with the tangent bundles of the foliations $\Psi_N(\mathcal{F}^{cu}_X)$, $\Psi_N(\mathcal{F}^u_X)$, $\mathcal{F}^{cs}_X$, and $\mathcal{F}^s_X$ respectively, on the fundamental domain $\bigcup_i X_{[0,N]}(T_i)$.

One can easily check the following classical result:

Lemma 7.2. *$\Psi_N \circ X_N$ is absolute partially hyperbolic if and only if*

$$\Psi_N(\mathcal{F}^u_X) \pitchfork \mathcal{F}^{cs}_X \quad \textit{and} \quad \Psi_N(\mathcal{F}^{cu}_X) \pitchfork \mathcal{F}^s_X.$$

Notice that $\{X_t(T_1)\}_{t\in\mathbb{R}}, \ldots, \{X_t(T_k)\}_{t\in\mathbb{R}}$ are pairwise disjoint, the same argument of Lemma 6.2 in [4] gives the following:

Lemma 7.3. *With the notation above, we have that for each $i = 1, \ldots, k$,*

$$\lim_{N\to+\infty} \psi_{i,N}(\mathcal{F}^{uu}_X) = \{\mathcal{F}^u_i\} \times \{s\} \quad \textit{and} \quad \lim_{N\to+\infty} \psi_{i,N}(\mathcal{F}^{ss}_X) = \{\mathcal{F}^s_i\} \times \{s\}$$

uniformly in the C^1-topology.

As a consequence of Lemma 7.3, when N is chosen large, for each $i = 1, \dots, k$, we have that

$$\Phi_i(\psi_{i,N}(\mathcal{F}_X^{uu})) \pitchfork \mathcal{F}_i^s \times [0,1] \quad \text{and} \quad \Phi_i(\psi_{i,N}(\mathcal{F}_X^{ss})) \pitchfork \mathcal{F}_i^u \times [0,1].$$

Now Theorem 7.1 follows directly from Lemma 7.2. □

Now, we end the proof of Theorem B by proving that the (absolute) partially hyperbolic diffeomorphism $f = \Psi_N \circ X_N$ is robustly dynamically coherent and plaque expansive. We denote by E^c_f the center bundle of f.

Recall that f coincides with X_N on the repelling region $X_{-N}(M^r)$ and on the attracting region M^a. Just as Lemma 9.1 in [4], we have that:

Lemma 7.4. *There exists a constant $C > 1$ such that for any unit vector $v \in E^c_f$, we have the following:*

$$\frac{1}{C} \leq \| Df^n(v) \| \leq C, \quad \textit{for any integer } n \in \mathbb{Z}.$$

As a consequence of Lemma 7.4, we have that f is *Lyapunov stable* and *Lyapunov unstable* in the directions E^{cs}_f and E^{cu}_f respectively.

To show the dynamically coherent and plaque expansive properties, we follow the same argument in [4, Theorem 9.4]:

- According to [17, Theorem 7.5], f is *dynamically coherent*, and center stable foliation $\mathcal{W}^{cs}_f$ and center unstable foliation $\mathcal{W}^{cu}_f$ are *plaque expansive*;
- By [16], the center stable foliation $\mathcal{W}^{cs}_f$ and center unstable foliation $\mathcal{W}^{cu}_f$ are *structurally stable*, proving that f is robustly dynamically coherent.

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