Axially symmetric polygons inscribed in and circumscribed about convex sets

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Axially Symmetric Polygons Inscribed in and Circumscribed about Convex sets[1])

In determining lower bounds for certain measures of central symmetry (centrality) for *ovals* – planar convex sets with non-void interior – it is useful to establish the existence of central polygons inscribed in (and circumscribed about) these sets [8]. Thus, e.g., Besicovitch [1] has shown the existence of an affine-regular hexagon inscribed in every oval with the added property that the ratio of their areas is never less than 2/3. In this paper we consider analogous questions with regard to axial symmetry (axiality). Some of these results are new, and others are collected together in this context for the first time. Many unsolved problems and conjectures still exist. For the application of these results to measures of axiality, see [3].

Since axiality is similarity-invariant, it is sufficient to consider ovals whose diameter is 1. In what follows, K shall denote an arbitrary oval, K_c a central oval, and K_1 an oval of constant breadth 1.

The interesting elementary axial polygons which can be inscribed in (circumscribed about) an oval are the isosceles triangle, kite (a quadrilateral symmetric about a diagonal), rhombus, rectangle, square, hexagon, and octagon. A polygon is ***properly*** inscribed in (circumscribed about) an oval K when each of its vertices (sides) is (contains) a point of βK, the boundary of K. We shall consider only properly inscribed (circumscribed) polygons, and shall discuss, in particular, the ratios of the areas and perimeters of these polygons of those of the given oval.

I. Area Ratios. The area of a set S is denoted by $[S]$.

A. *The Isosceles Triangle.*

Theorem 1. In every oval K there is an isosceles triangle $I'(K)$ such that

$$\frac{[I'(K)]}{[K]} \geqslant \frac{3}{8}\,.$$

Proof. It is sufficient to prove the theorem for strictly convex ovals, since an arbitrary oval can be realized as the limit of an appropriate sequence of strictly

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convex ovals, and since the limit of a sequence of isosceles triangles is again an isosceles triangle. For every direction φ in the plane, there is, among all the triangles inscribed in an oval K with a side in this direction, at least one, denoted T_φ, which has maximal area. There is also a (not necessarily unique) triangle $T = A_1 A_2 A_3$ of maximal area among all the triangles T_φ. If T is isosceles, for a particular K, then the theorem follows from a result of BLASCHKE [2, p. 50], which states that in every oval K there is a triangle $T(K)$ (though not necessarily isosceles in general) such that $[T(K)]/[K] \geqslant 3\sqrt{3}/4\pi \sim 0.413$, with equality if and only if K is an ellipse. We may assume, then, that T is not isosceles. Let a_i, $i = 1, 2, 3$, denote the side of T opposite A_i, H_i the foot of the altitude to a_i (extended if necessary), and a_i', a_a'' the segments $A_{i+1} H_i$, $H_i A_{i+2}$, respectively, where the addition in the indices is modulo 3. Clearly, $a_i'/a_a'' = \cot A_{i+1}/\cot A_{i+2}$, and since T is not isosceles, none of the ratios a_i'/a_i'' has the value 1. We may assume, with no loss in generality, that $a_1'/a_1'' > 1$; then at least one of the other ratios, say a_2'/a_2'', is less than 1 (for the product of the three ratios is 1). Let φ_i be the direction of a_i. Since T is a triangle of maximal area in K, $T_{\varphi_i} = T$, $i = 1, 2, 3$. Under the hypothesis of strict convexity, the ratios a_i'/a_i'' are continuous functions of φ. Let A be the vertex of T_φ opposite the side a lying in the direction φ, and define $f(\varphi) = a'/a''$ as above. As φ varies from φ_1 to φ_2, $f(\varphi)$ varies from $a_1'/a_1'' > 1$ to $a_2'/a_2'' < 1$, so that for some direction $\varphi_0 \in (\varphi_1, \varphi_2)$, $f(\varphi_0) = 1$, i.e. T_{φ_0}. is isosceles. HODGES [10] has shown that $\inf_\varphi\{[T_\varphi]/[K]\} \geqslant 3/8$, so that this inequality must also be true for the isosceles triangle T_{φ_0}. This proves the theorem.

It is doubtful that the bound 3/8 is the best possible. We conjecture, rather, that the result of BLASCHKE, mentioned above, remains true under the restriction that the inscribed triangle $T(K)$ is isosceles, and thus provides the greatest lower bound in this case. Since the circle is both central and an oval of constant breadth, the result of BLASCHKE proves that no greater bound can be conjectured for this area ratio when the oval K is central or an oval of constant breadth. However, EGGLESTON and TAYLOR [7] have shown that in every oval K_1 of constant breadth 1 there is an inscribed equilateral triangle $\Delta'(K_1)$ of side length at least $\lambda \sim 0.8534$, with equality for the 'REULEAUX pentagon' P_1, and since the circle C_1 is the curve of constant breadth 1 enclosing the greatest area, we have the improved inequality

$$\frac{[I'(K_1)]}{[K_1]} \geqslant \frac{[\Delta'(K_1)]}{[K_1]} \geqslant \frac{[\Delta'(K_1)]}{[C_1]} \geqslant \frac{[\Delta'(P_1)]}{[C_1]} \sim 0.4015\,, \tag{1}$$

for ovals of constant breadth.

Theorem 2. About every oval K, there is a circumscribed isosceles triangle $I''(K)$ such that $[K]/[I''(K)] \geqslant 1/2$, with equality for parallelograms.

Proof. EGGLESTON [5] has established the 'dual' of the result of HODGES mentioned above, viz. that if T_φ is a triangle of minimal area circumscribed about an oval K with one side in the direction φ, then $\inf_\varphi\{[K]/[T_\varphi]\} \geqslant 1/2$, and that $[K]/[T_\varphi] = 1/2$ for every direction φ only when K is a parallelogram. Therefore, it is sufficient to show that for at least one direction φ_0, the triangle T_φ. is isosceles. The proof of this fact is exactly like that of Theorem 1, and will be omitted.

Since parallelograms are central, no greater bound for this ratio is possible when K is restricted to the class of central ovals. It is well known (HADWIGER, DEBRUNNER, KLEE [9, p. 14]), that every oval of diameter 1 is contained in an equilateral triangle Δ'' of side length $\mu = \sqrt{3}$, with equality for C_1. Since the REULEAUX triangle T_1 has least area among all ovals of constant breadth 1, it follows from these facts that

$$\frac{[K_1]}{[I''(K_1)]} \geqslant \frac{[K_1]}{[\Delta'']} \geqslant \frac{[T_1]}{[\Delta'']} = \frac{2}{9}\left(\pi\sqrt{3} - 3\right) \sim 0.5426 \tag{2}$$

is a better inequality for this ratio on the class of curves of constant breadth. We conjecture that

$$\frac{[K_1]}{[I''(K_1)]} \geqslant \frac{[C_1]}{[I''(C_1)]} = \frac{[C_1]}{[\Delta'']} = \frac{\pi}{3\sqrt{3}} \sim 0.6046 \tag{3}$$

is the best possible inequality on this class, with equality only for C_1.

B. *The Kite.* A *diameter* of K is a chord of maximal length and a *width* is a chord of minimal length.

Lemma 1. A diameter and a width of an oval K cannot both be subsets of βK.

The proof of this lemma is straightforward, and will be omitted.

Theorem 3. In every oval K, there is an inscribed kite $Q'(K)$ such that $[Q'(K)]/[K] \geqslant 1/2$.

Proof. Assume, without loss of generality, that a diameter AB of K does not lie in βK, and draw (parallel) support lines to K through the points A and B. Let CD be the chord of K on the perpendicular bisector of AB, and draw support lines to K through C and D. These four support lines determine a trapezoid circumscribed about K. The quadrilateral $ACBD$ (which is not degenerate by the above assumption) is symmetric with respect to its diagonal CD, and clearly has area half that of the circumscribed trapezoid, so that the desired inequality follows. That this bound cannot be improved may be seen by choosing for K a 'thin' rectangle. It is not known whether the bound is realized for any oval.

Since a rectangle is central, this same bound is also the best possible on the subclass of central ovals. However, for ovals of constant breadth, we have a better result.

Let $S = ABCD$ be a unit square circumscribed about an oval K_1 of constant breadth 1, E and F the midpoints of AB and CD, respectively, and ζ the closed (shaded) region in Fig. 1, constructed by drawing circular arcs of radius 1 about E and F, and about the intersections of these arcs with the other two sides of S. We may rotate K_1 inside S so that E and F belong to βK_1. For this position of K_1, which we shall call a *standard position* of K_1 with respect to S, we have the following

Lemma 2. $\beta K_1 \subseteqq \zeta$.

This lemma is a direct consequence of simple properties of ovals of constant breadth.

Theorem 4. $[Q'(K_1)]/[K_1] \geqslant 2/\pi \sim 0.6366$, with equality only for C_1.

Proof. This theorem is an immediate consequence of the preceding lemma and of the isoperimetric inequality.

Concerning circumscribed kites of minimal area, nothing is known except for ovals of constant breadth, where, if we consider the circumscribed unit square S as a kite, we find that

$$\frac{[K_1]}{[Q''(K_1)]} \geqslant \frac{[K_1]}{[S]} \geqslant \frac{[T_1]}{[S]} = \frac{1}{2}\left(\pi - \sqrt{3}\right) \sim 0.7048\,. \tag{4}$$

We conjecture that the best possible result for ovals of constant breadth is

$$\frac{[K_1]}{[Q''(K_1)]} \geqslant \frac{[C_1]}{[Q''(C_1)]} = \frac{[C_1]}{[S]} = \frac{\pi}{4} \sim 0.7854\,, \tag{5}$$

with equality only for C_1.

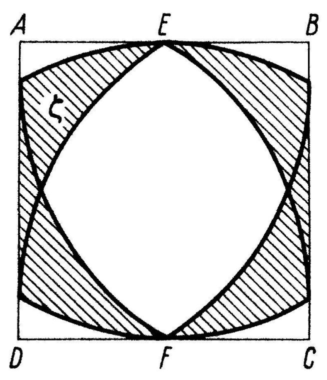
C. *The Rhombus.* Nothing is known here regarding arbitrary ovals. For central ovals, the inscribed kite of Theorem 3 is always a rhombus, so that the same bound holds in this case.

Theorem 5. In every oval K_1 of constant breadth 1, there is an inscribed rhombus $R'(K_1)$ with a diagonal of length 1 such that $[R'(K_1)]/[K_1] \geqslant \alpha_0 \sim 0.5483$.

Proof. If $S = ABCD$ is a unit square circumscribed to K_1, we can rotate K_1 through an angle φ inside S such that the points of contact $E' = E(\varphi)$ and $F' = F(\varphi)$ of βK_1 with AD and BC, respectively, make the distance $DE' = CF'$ a minimum. If $Y' = Y(\varphi)$ and $Z' = Z(\varphi)$ are the points of contact of the perpendicular bisector of $E'F'$ with βK_1, we may select these such that $Y'X' \geqslant Z'X'$, where $X' = E'F' \cap Y'Z'$. If $Y'X' = Z'X'$, then $E'Z'F'Y'$ is the desired rhombus, and we are done. Assume, then, that $Y'X' > Z'X'$. Rotating K_1 in S through an angle π from φ to $\varphi + \pi$, we obtain $E'' = E(\varphi + \pi) = AD \cap \beta K_1$, $F'' = F(\varphi + \pi) = BC \cap \beta K_1$ and $AE'' = DE' = BF'' = CF'$. $Y'' = Y(\varphi + \pi)$ and $Z'' = Z(\varphi + \pi)$ are defined analogously to Y' and Z', so that $Y''X'' = Z'X'$ and $Z''X'' = Y'X'$, where $X'' = E''F'' \cap Y''Z''$. Since βK_1 is a continuous curve, the length $Y'X' - Z'X' = f(\varphi)$ is a continuous function of φ. The above remarks show that $Y'X' - Z'X' = f(\varphi) > 0 > f(\varphi + \pi) = Y''X'' - Z''X''$, so that there is a φ_0 in the interval $(\varphi, \varphi + \pi)$ such that $f(\varphi_0) = 0$, which implies that the quadrilateral $EZFY$ for this position φ_0 is a rhombus, since its diagonals are mutual perpendicular bisectors. Since $EF = 1$, the existence of the desired rhombus is established.

Referring to Lemma 2 and Fig. 1, an easy computation shows that the shorter diagonal of the inscribed rhombus has minimum length $d = 2\sqrt{\{1 - (\sqrt{3} - 1)^2/4\}} - 1$ for the Reuleaux triangle T_1. Therefore,

$$\frac{[R'(K_1)]}{[K_1]} \geqslant \frac{[R'(T_1)]}{[K_1]} \geqslant \frac{[R'(T_1)]}{[C_1]} = \frac{2\,d}{\pi} = \alpha_0 \sim 0.5483\,.$$



Figur 1

We conjecture that the best possible inequality in this case is

$$\frac{[R'(K_1)]}{[K_1]} \geqslant \frac{[R'(T_1)]}{[T_1]} = \frac{d}{\pi - \sqrt{3}} \sim 0.611\,, \tag{6}$$

with equality only for T_1.

D. *The Rectangle.* RADZISZEWSKI [12] has shown the existence, in every oval K, of an inscribed rectangle of area at least half that of K. That this bound cannot be improved for central ovals may be seen by taking for K a rhombus with an angle $\alpha < \pi/4$. We conjecture that, in ovals of constant breadth, inscribed rectangles of maximal area are squares.

It is easy to see that, about every oval K, a rectangle may be circumscribed with area no greater than twice that of K. For this purpose, it is sufficient to take a side of the rectangle in the direction of a diameter of K. By considering a 'thin' rhombus with an angle nearly zero, we see that this bound cannot be improved on central ovals. For ovals of constant breadth, every circumscribed rectangle is a square.

E. *The Square.* Inscribed and circumscribed squares are of interest only for ovals of constant breadth, for with arbitrary and central ovals the area ratio may be near zero. EGGLESTON [6] established the existence, in every K_1, of an inscribed square $S'(K_1)$ of edge length at least $\sigma \sim 0.6474$, with equality only for T_1. Hence,

$$\frac{[S'(K_1)]}{[K_1]} \geqslant \frac{[S'(T_1)]}{[K_1]} \geqslant \frac{[S'(T_1)]}{[C_1]} = \frac{4\,\sigma^2}{\pi} \sim 0.5336\,. \tag{7}$$

We conjecture that

$$\frac{[S'(K_1)]}{[K_1]} \geqslant \frac{[S'(T_1)]}{[T_1]} = \frac{2\,\sigma^2}{\pi - \sqrt{3}} \sim 0.5945\,. \tag{8}$$

Since the unit square S is the only one which may be circumscribed about any K_1, we clearly have

$$\frac{[K_1]}{[S]} \geqslant \frac{[T_1]}{[S]} = \frac{1}{2}\left(\pi - \sqrt{3}\right) \sim 0.7048\,. \tag{9}$$

F. *The Hexagon.* The set of midpoints of all chords of an oval K in a fixed direction φ is called the *load curve* ('Schwerlinie') of K in the direction φ, and denoted $\lambda_\varphi(K)$. The basic facts about load curves are contained in a paper of ZINDLER [14]. Here we make use of the simple observations that every load curve of an oval is connected, and that in the direction φ_d normal to a diameter d of K, the endpoints of $\lambda_{\varphi_d}(K)$ are also endpoints of this diameter. This last remark follows from the well known fact that the support lines to an oval K in the direction φ_d meet $\beta\, K$ in only one point, viz. an endpoint of the diameter d.

Theorem 6. In every oval there is an inscribed axial hexagon.

Proof. Let A and B be endpoints of a diameter of K, and φ_d the direction normal to AB. Since A and B are also endpoints of the load curve $\lambda_{\varphi_d}(K)$, which is a connected set, there is at least one line k in the direction parallel to AB which intersects λ_{φ_d} in two separate points distinct from A and B. (This line could contain the segment AB, if $\lambda_{\varphi_d} = AB$.) These two points in $\lambda_{\varphi_d} \cap k$ determine two chords of K in the direction φ_d which are bisected by k; their four endpoints, together with the two points $k \cap \beta K$, form the vertices of a hexagon inscribed in K and symmetric with respect to k. (Clearly, such a hexagon need not be unique.)

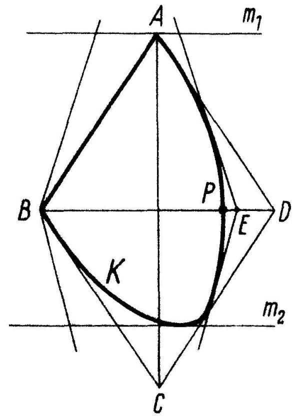
No non-trivial area ratios are known for inscribed and circumscribed hexagons.

Theorem 7. About every oval there is a circumscribed axial hexagon. (It is sufficient to prove the theorem for regular ovals, since the general case can then be obtained by approximation. We prefer, however, to give a constructive proof for the general case since a too hasty application of the approximation principle sometimes leads to errors. For example, the proof in YAGLOM-BOLTYANSKIJ [13, p. 144] that an *equiangular* hexagon having an axis of symmetry can be properly circumscribed about every oval is correct only for regular ovals, since, in particular, the statement is not true for a triangle with an angle $> 2\pi/3$.)

Proof. In case K is regular, the hexagon may be obtained from a circumscribed rhombus (which always exists, by an easy application of Bolzano's theorem) by snipping off opposite corners with support lines to K parallel to the diagonal through the other two vertices. Difficulties arise if K is not regular, for then support lines may pass through the vertices of the circumscribed rhombus. We distinguish several cases.

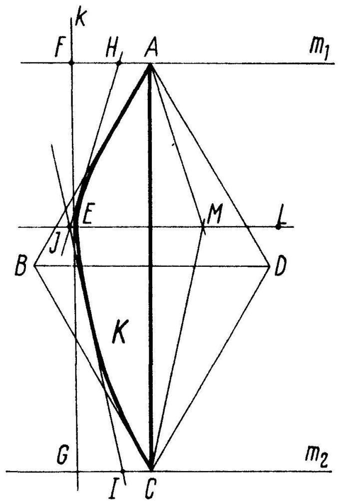
Case I. All four vertices of the circumscribed rhombus R are points of βK. In this case R coincides with K. At a pair of opposite vertices of K draw segments whose midpoints are these vertices, and which are parallel to and of smaller length than the diagonal joining the other pair of vertices. The endpoints of these two segments and of this diagonal are vertices of the desired hexagon.

Case II. A pair of adjacent vertices of R belong to βK. Let A and B be these two vertices, and AC, BD the diagonals of R. We may assume that neither C nor D is a point of βK, for if both are, then this case reduces to the preceding case, and if one is, to the following. Let $P \in BD \cap \beta K$, $P \neq B$, and E any point of the open segment PD^0 (Fig. 2). Draw support lines m_1 and m_2 to K parallel to BD. Since $A \in \beta K$, one of these lines, say m_1, passes through A; m_2 does not pass through C, by hypothesis. Through E draw two support lines to K; these are unique, once E is fixed. Through B draw two support lines to K making, with m_1 and m_2, respectively, angles equal to those made with these two lines by the support lines through E. These six support lines determine the desired hexagon.



Figur 2

Case III. A pair of opposite vertices of R belong to βK. Let A and C be these vertices and assume that K lies entirely to one side of the diagonal AC (Fig. 3). By convexity, $AC \subseteqq \beta K$. Draw support lines m_1 and m_2 through A and C, respectively,



Figur 3

and parallel to BD. Let k be the support line to K parallel to, and distinct from AC, and let $F = k \cap m_1$, $G = k \cap m_2$. Choose any two points $H \in AF$ and $I \in CG$ such that $AH = CI$, and draw the support lines to K through H and I meeting in J; finally, construct $JL \parallel BD$ and select the point $M \in JL$ such that $\triangle JHI \cong \triangle MAC$, and M is on the same side of AC as D. Then $AMCIJH$ is the desired hexagon.

If K does not lie to one side of AC, or if only one vertex of R belongs to βK, then an axial hexagon is easily constructed as in the case when K is regular. Since this exhausts all possibilities, the theorem is proved in general.

About every oval of constant breadth 1, there is circumscribed a regular hexagon H of edge length $\sqrt{3}/3$ (EGGLESTON [4, p. 127]), so that if $H''(K_1)$ denotes the circumscribed axial hexagon of minimal area, we have

$$\frac{[K_1]}{[H''(K_1)]} \geqslant \frac{[K_1]}{[H]} \geqslant \frac{[T_1]}{[H]} = \frac{1}{3}\left(\pi\sqrt{3} - 3\right) \sim 0.8137 . \tag{10}$$

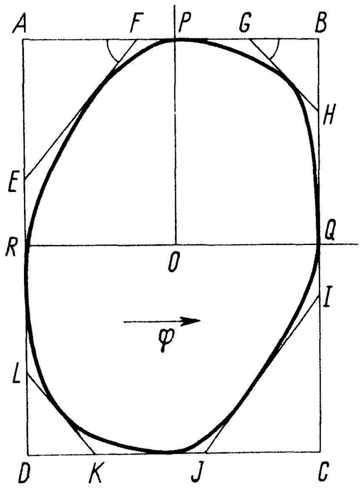
G. *The Octagon.* Properly inscribed and circumscribed octagons do not exist for every oval (e.g. the triangle). However, for a central oval K_c, a result of NOHL [11] shows the existence of an inscribed axial (and central) octagon $O'(K_c)$ such that

$$\frac{[O'(K_c)]}{[K_c]} \geqslant 2\left(\sqrt{2} - 1\right) \sim 0.828, \tag{11}$$

with equality for a certain class of parallelograms. A 'dual' result is the following

Theorem 8. About every central oval K_c, there is a circumscribed axial (and central) octagon $O''(K_c)$ such that $[K_c]/[O''(K_c)] \geqslant \sqrt{2}/2 \sim 0.707$.

Proof. No difficulty arises in this case in proving the theorem for regular central ovals, and then using an approximation argument to obtain full generality. For every direction φ in the plane, let R_φ be a rectangle circumscribed about K_c with a pair of sides in this direction. Since K_c is central, its center O coincides with that of R_φ. Let AB and CD be the sides of R_φ in the direction φ. Since K_c is regular, by hypothesis, the vertices of R_φ do not belong to βK_c, so we may draw support lines EF and GH to K_c making equal angles with AB, as in Fig. 4. Let P, Q be the midpoints of AB,



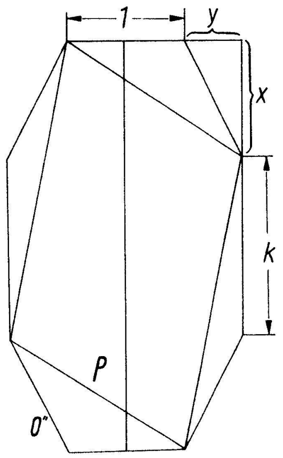
Figur 4

BC, respectively. If $FP = PG$, then the octagon $O'' = EFGHIJKL$ (where I, J, K, L are images, respectively, of the points E, F, G, H reflected in O) is axial, and the theorem is proved. Suppose that this is not the case, but that, in particular, $FP < PG$; then $AF > GB$, $AE > BH$, and $ER = QI < QH$. Define $a(\varphi) = FP$ and $b(\varphi) = PG$; then $a(\varphi + \pi/2) = HQ$ and $b(\varphi + \pi/2) = QI$. The above inequalities imply that $a(\varphi) - b(\varphi) < 0$ and $a(\varphi + \pi/2) - b(\varphi + \pi/2) > 0$, and since it is geometrically obvious that the function $f(\varphi) = a(\varphi) - b(\varphi)$ is continuous, it follows that for some intermediate direction φ_0 the circumscribed octagon O_{φ_0} has an axis of symmetry.

A unique (up to congruence) parallelogram P may be properly inscribed in an axial and central octagon O'' by joining its alternate vertices. If O'' is circumscribed about a central oval K_c, we now show that $[K_c]/[O''] \geqslant [P]/[O'']$. For this purpose, let A', B', C', D', E', F', G', H' be the points of contact of βK_c with O'', selected so that $B' \in AB$, $C' \in BC, \dots, A' \in HA$ (two of these may coincide at a vertex of O'', but this does not affect the proof), and let O' denote the octagon determined by them. Since $O' \subseteqq K_c$, $[O'] \leqslant [K_c]$. The vertices of O' and of O'' are symmetric with respect to the center O of K_c. We now show that $[O'] \geqslant [P]$.

Either $BB' \geqslant CD'$ or $BB' \leqslant CD'$; assume the latter (with no loss in generality); then $[B'C'D'] \geqslant [B'BD']$ and (by symmetry) $[F'G'H'] \geqslant [F'FH']$, so that $[O'] \geqslant [A'B'BD'E'F'FH'] \geqslant [A'BD'E'FH']$. Now $BD'E'F = BD'E' \cup E'FB$ and either $[BD'E'] \geqslant [BCE']$ or $[BD'E'] \geqslant [BDE']$. Suppose the former; then $[BD'E'F] = [BD'E'] + [E'FB] \geqslant [BCE'] + [E'FB] = [BCE'F] \geqslant [BDF]$; on the other hand, if the latter inequality is true, then $[BD'E'F] = [BD'E'] + [E'FB] \geqslant [BDE'] + [E'FB] = [BDE'F] \geqslant [BDF]$. By symmetry, we also have $[FH'A'B] \geqslant [FHB]$, so that $[A'BD'E'FH'] = [BD'E'F] + [FH'A'B] \geqslant [BDF] + [FHB] = [BDFH] = [P]$, which implies, by a previous inequality, that $[O'] \geqslant [P]$.

We now determine the minimum value of the ratio $[P]/[O'']$. We may assume, without loss of generality, that the lengths of two sides of O'' which lie along the edges of the circumscribed rectangle R are 1 and k, where $0 < k < \infty$. Then, referring to Fig. 5, $[P] = k + ky + x + 2xy$, $[O''] = k + 2ky + 2x + 2xy$, and the function $f(k, x, y) = [P]/[O'']$ has the unique minimum value of $\sqrt{2}/2$ when $x = y = \sqrt{2}/2$ and $k = 1$, as can be shown by elementary calculus. In this case, O'' is a regular octagon, and P is a square. Therefore, $[K_c]/[O''] \geqslant [O']/[O''] \geqslant [P]/[O''] \geqslant \sqrt{2}/2$.



Figur 5

II. Perimeter Ratios. The perimeter of a set S is denoted $|S|$. The known results here are rather meager, and most of these concern ovals of constant breadth.

A. *The Isosceles Triangle.* Using the result of EGGLESTON and TAYLOR mentioned on p. 3, and the notation of the preceding section, we can conclude immediately, as regards ovals of constant breadth, that

$$\frac{|\Gamma'(K_1)|}{|K_1|} \geqslant \frac{|\Delta'(K_1)|}{|K_1|} \geqslant \frac{|\Delta'(P_1)|}{|K_1|} = \frac{3\lambda}{\pi} \sim 0.8149, \tag{12}$$

since it is well known that every oval of constant breadth 1 has perimeter π.

As far as circumscribed isosceles triangles are concerned, the best possible result is known and follows from a theorem of EGGLESTON [5]: About every oval K there is a circumscribed equilateral triangle $\Delta''(K)$ such that

$$\frac{|K|}{|\Delta''(K)|} \geqslant \frac{\pi}{3\sqrt{3}} \sim 0.6046, \tag{13}$$

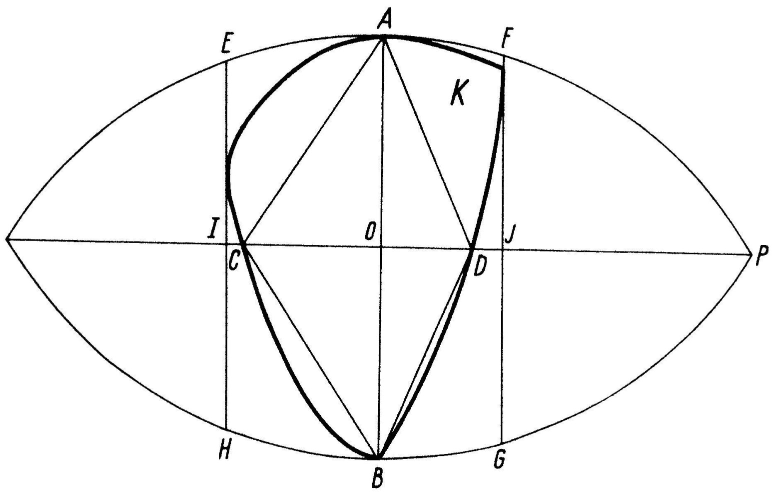
with equality only for the circle.

B. *The Kite.*

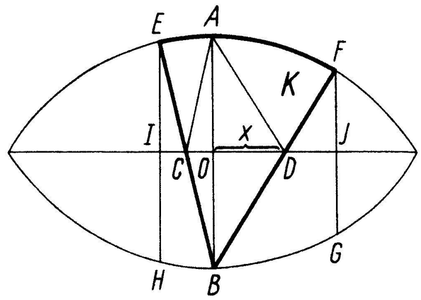
Theorem 9. In every oval K, there is an inscribed kite $Q'(K)$ such that $|Q'(K)|/|K| \geqslant \beta_0 - \varepsilon \sim 0.649 - \varepsilon$, where $\varepsilon > 0$ is arbitrarily small.

Proof. With $AB = 1$ a diameter of K, K can be covered by a lens-shaped region (Fig. 6) bounded by circular arcs of radius 1. The points A, B and the points of intersection C and D of βK with the perpendicular bisector OP of AB are vertices of a kite inscribed in K. (The case where $AB \subseteqq \beta K$ and the kite degenerates to an isosceles triangle does not affect the first part of the proof, and will be dealt with later.) Draw support lines EH and FG to K parallel to AB. Then, if $Q' = ACBD$ and $K' = EFGH$, $Q' \subseteqq K \subseteqq K'$ implies $|Q'|/|K| \geqslant |Q'|/|K'|$. For a fixed Q', the minimum value of the ratio $|Q'|/|K'|$ occurs for a maximum value of the arc length $\widehat{EF}$, which in turn occurs when a support line to K through C (respectively D) coincides with either AC

or BC (respectively AD or BD), as, for example, is the case when K is a circular sector of radius 1 (see Fig. 7).



Figur 6



Figur 7

With B the origin of a rectangular coordinate system, referring to Fig. 7, we set $OD = x$ and define $a = AD + BD = \sqrt{\{4x^2 + 1\}}$; $b = \widehat{AF} + FG + \widehat{BG} = 2\arctan(2x) + 2/\sqrt{\{4x^2 + 1\}} - 2$; $c = AC + CB$; $d = \widehat{AE} + \widehat{EH} + HB$. By symmetry, it is sufficient to determine the minimum value of only one of the ratios a/b and c/d. For this purpose we define $f(x) = a/b$; this function has a unique minimum value $\beta_0 \sim 0.649$ on the interval $[0, \sqrt{3}/2]$ achieved for $x = x_0 \sim 0.28$. From this it follows that

$$\frac{|Q'(K)|}{|K|} \geqslant \frac{|Q'(K)|}{|K'|} \geqslant \frac{|ACBD|}{|EFGH|} = \frac{a+c}{b+d} \geqslant \min\{a/b, c/d\} = \min f(x) = \beta_0,$$

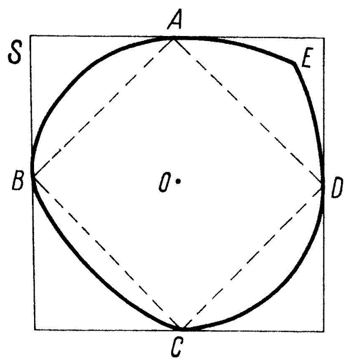
which proves the theorem, provided Q' is not degenerate.

If Q' degenerates to an isosceles triangle for some oval K, it is easy to construct a proper kite Q'' inscribed in K such that $|Q''| \geqslant |Q'| - \epsilon$ for every $\epsilon > 0$, by constructing one diagonal of Q'' parallel to, and arbitrarily close to, the diameter AB, but lying (except for its endpoints) entirely in the interior of K. The theorem is thus proved in general.

For a result concerning central ovals, we defer to the next section.

Theorem 10. For ovals of constant breadth, $|Q'(K_1)|/|K_1| \geqslant 2\sqrt{2}/\pi \sim 0.9002$, with equality only for those ovals having two diameters which are mutual perpendicular bisectors.

Proof. With K_1 in standard position with respect to the circumscribed square S, the points of contact of βK_1 with S are vertices of an inscribed kite. It is well known that, among all triangles with fixed base and area, the isosceles triangle on this base has least perimeter. Therefore, this inscribed kite has minimum perimeter when it is a square, and the theorem follows easily. An example of an oval of constant breadth which is not a circle and which admits an inscribed square of diagonal 1 is given in Fig. 8, and is constructed by drawing circular arcs $\widehat{AB}$ and $\widehat{CD}$ of radius 1/2 about the center O of the circumscribed square S, and circular arcs $\widehat{AE}$, $\widehat{ED}$, and $\widehat{BC}$ of radius 1 about the points C, B and E, where A, B, C, and D are midpoints of the sides of S.

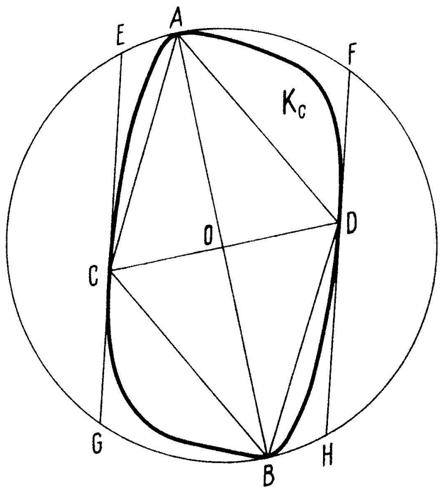


Figur 8

C. *The Rhombus.* Nothing is known in this case for arbitrary ovals.

Theorem 11. In every central oval K_c, there is an inscribed rhombus $R'(K_c)$ such that $|R'(K_c)|/|K_c| \geqslant \gamma_0 \sim 0.8045$.

Proof. Every central oval K_c of diameter 1 is contained in a circle of diameter 1 with the same center O. Let AB be a diameter of K_c, CD the chord through O which is perpendicular to AB, and EG, FH support lines to K_c through C and D, respectively



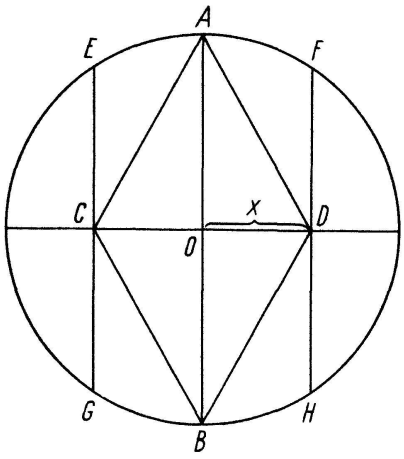
Figur 9

(Fig. 9). Since K_c is central, $EG \parallel FH$, and the quadrilateral $ADBC$ is a rhombus. The perimeter of this rhombus is smallest, for a fixed circumscribing figure $K' = EFHG$, when its diagonal CD is a minimum, which occurs when $EG \perp CD$. Let $R = R(K')$ denote this minimum rhombus.

If O is the origin of a rectangular coordinate system, then (see Fig. 10) we have $a = AD + DB = \sqrt{\{4x^2 + 1\}}$ and $b = \widehat{AF} + FH + \widehat{HB} = \arcsin(2x) + \sqrt{\{1 - 4x^2\}}$. Defining $g(x) = a/b$, we find that g has a unique minimum value $\gamma_0 \sim 0.8045$ on the interval $[0, 1/2]$ attained for $x = x_0 \sim 0.2565$. Therefore

$$\frac{|R'(K_c)|}{|K_c|} \geqslant \frac{|R'(K_c)|}{|K'|} \geqslant \frac{|R|}{|K'|} = \frac{a}{b} \geqslant \min g(x) = \gamma_0 .$$

As an analogue of Theorem 5 we have the following



Figur 10

Theorem 12. In every oval of constant breadth 1 there is an inscribed rhombus $R'(K_1)$ with a diagonal of length 1 such that $|R'(K_1)|/|K_1| \geqslant \delta_0 \sim 0.8402$, with equality for the REULEAUX triangle T_1.

Proof. The existence of $R'(K_1)$ has been established in Theorem 5, from which it also follows that $|R'(K_1)|$ has the minimum value $p = 4\left\{1/4 + \left(\sqrt{[\sqrt{3}/2]} - 1/2\right)^2\right\}$, as an easy computation shows, when K_1 is the REULEAUX triangle. Therefore,

$$\frac{|R'(K_1)|}{|K_1|} \geqslant \frac{|R'(T_1)|}{|K_1|} = \frac{p}{\pi} = \delta_0 \sim 0.8402\,.$$

D. *The Rectangle.* No results are known in this connection for rectangles other than squares.

E. *The Square.* As was pointed out in the previous section, the only interesting case is that of ovals of constant breadth. Using the result of EGGLESTON [6] mentioned on p. 7, and the notation of that paragraph, we can conclude immediately that

$$\frac{|S'(K_1)|}{|K_1|} \geqslant \frac{|S'(T_1)|}{|K_1|} = \frac{4\,\sigma}{\pi} \sim 0.8243\,, \tag{14}$$

with equality for T_1.

The analogous result for circumscribed squares is obvious.

F. *The Hexagon.* Since there is a regular hexagon H of side length $\sqrt{3}/3$ circumscribed about every oval of constant breadth 1, if $H''(K_1)$ denotes the circumscribed axial hexagon of minimal perimeter, we have

$$\frac{|K_1|}{|H''(K_1)|} \geqslant \frac{|K_1|}{|H|} = \frac{\pi}{2\sqrt{3}} \sim 0.9069\,. \tag{15}$$

G. *The Octagon.* No perimeter ratios are known for inscribed or circumscribed axial octagons.

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Kleine Mitteilungen

Bestimmung einer oberen Schranke für den Inhalt des Parallelrisses eines regelmässigen Körpers

1. Im dreidimensionalen euklidischen Raum E_3 gibt es bekanntlich fünf regelmässige, konvexe oder PLATONische Körper. Ist s eine beliebige Sehstrahlrichtung und π eine zu s normale Bildebene, so ist der scheinbare Umriss eines solchen PLATONischen Körpers Π ein ebenes konvexes Polygon Π^n, das im folgenden kurz als der Normalriss von Π bezeichnet werden soll; F sei sein Flächeninhalt. Das PLATONische Polyeder Π habe p (untereinander kongruente und regelmässige) Polygone zu Seitenflächen s_i $(i = 1, 2, \dots p)$, die alle denselben Flächeninhalt f besitzen. Ist ε_i die Trägerebene von s_i und α_i der Neigungswinkel von ε_i gegen die Bildebene π, so hat der Normalriss s_i^n von s_i einen durch

$$f_i = f \cos\alpha_i \tag{1}$$

gegebenen Flächeninhalt. Da jeder Sehstrahl s, der mit dem konvexen Polyeder Π innere Punkte gemeinsam hat, den Rand $\mathfrak{R}(\Pi)$ von Π in genau zwei Punkten trifft, ist jeder innere Punkt des Normalrisses Π^n von Π Normalriss von genau zwei Punkten des Randes $\mathfrak{R}(\Pi)$ von Π und es gilt mithin für den Flächeninhalt F von Π^n:

$$2\,F = \sum_{i=1}^{p} f_i\,. \tag{2}$$

Ist O der Mittelpunkt des PLATONischen Körpers und O_i der Mittelpunkt seiner Seitenfläche s_i, so ordnen wir jeder Seitenfläche s_i einen zum Vektor $\overrightarrow{OO_i}$ proportionalen Vektor $\mathfrak{m}_i$ zu, dessen Betrag mit dem Inhalt f von s_i übereinstimmt. Jeder Vektor $\mathfrak{m}_i$ steht somit auf der Trägerebene ε_i von s_i normal; die Spitzen der Vektoren $\mathfrak{m}_i$ bestimmen in ihrer Gesamtheit als von O ausgehende Ortsvektoren ein zu dem gegebenen PLATONischen Polyeder Π «duales» regelmässiges Polyeder.