Theorems related to Wallace's (Simson's) Line

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Part I. A «dual» Theorem

If P, Q, R are points where a tangent to the inscribed circle of a triangle ABC meets the diameters of this circle which are perpendicular to the bisectors of the angles at A, B, C, respectively, then the lines AP, BQ and CR are concurrent.

The very close relation of this theorem to Wallace's theorem can be made quite clear when both theorems are generalized, i.e. stated in terms of projective geometry. It is then seen that each theorem is the dual of the other.

Below we list the steps which lead to the generalised theorem of Wallace and the dual steps to the theorem stated above.

1. In the plane let there be a line – l_∞ – and on that line two (isotropic) points I and J.

1a. In the plane let there be a point – O – and through that point two (isotropic) lines i and j.

2. Let there further be three points A, B, C so that no three of the points A, B, C, I, J are collinear.

2a. Let there further be three lines a, b, c so that no three of the lines a, b, c, i, j are concurrent.

3. There is one conic through A B C I J viz. the circumcircle of the triangle ABC.

3a. There is one conic touching a, b, c, i, j. This is an ellipse or hyperbola with focus O; if the polar of O with respect to the conic is taken as l_∞, the conic is either the inscribed circle or one of the escribed circles of the triangle a, b, c.

4. Let T be a point on the conic. Draw three lines through T at right angles to BC, CA and AB, i.e. join T to points on l_∞ which are separated harmonically from the points in which l_∞ is cut by BC, CA, AB respectively by the isotropic points I and J.

4a. Let t be a tangent to the conic. Cut t with three lines at right angles to OA, OB, OC, where A is the intersection of b and c, etc.; i.e. cut t with three lines through O which are separated harmonically from lines joining O with the intersection of (bc), (ca) and (ab) respectively by the isotropic lines i and j.

5. Cut these three lines through T by the lines BC, CA, AB respectively. The three intersections lie on a straight line w. (Generalized Theorem of Wallace.)

This juxtaposition not only leads to the theorem stated above but at the same time constitutes a proof of this theorem.

It is interesting to continue the comparison of the two cases a bit further.

6. If the point T moves along the conic the line w will envelop a curve which because of its one to one correspondence with a conic must be of genus 0, too.

6a. If t moves along the conic the point W will describe a curve of genus 0.

7. It is evident that the sides and altitudes of $\triangle$ ABC must be tangents to the curve, which therefore must be of class three at least.

From Figure 1 it is evident that to each point P on the line AB there are two and only two points P' and P'' on the circumcircle for which the Wallace line passes through P and is distinct from AB; i.e. there are three and only three Wallace lines

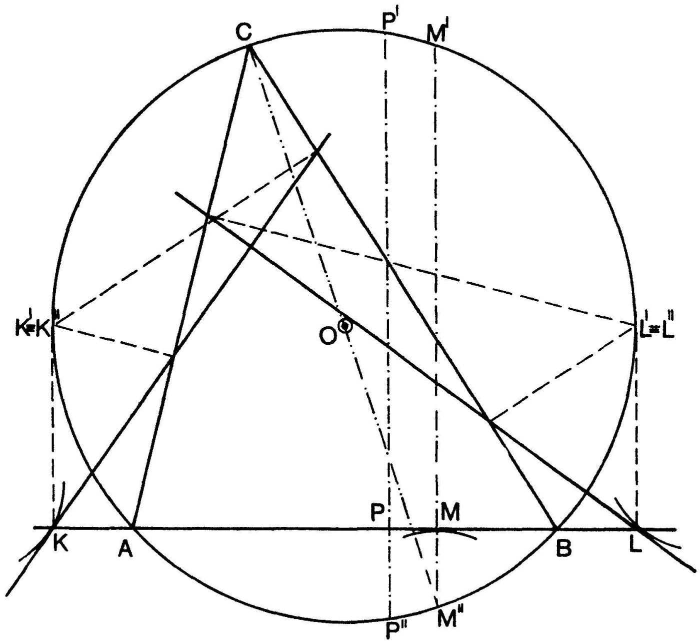

Figure 1

passing through P. If P coincides with K or L the points K' and K'' respectively L' and L'' coincide and the corresponding Wallace line touches the envelope at K respectively L. If P coincides with M one of the corresponding Wallace lines is the line AB; therefore AB touches the envelope at M.

A curve of genus 0 and of class 3 must be of order 4. We will refer to the envelope as C_3^4.

7a. In the dual case a similar reasoning leads us to a curve of class 4 and order 3: Γ_4^3.

8. A curve of genus 0 and order 4 must have 3 cusps or nodal points. However nodal points are incompatible with a class lower than 4. Hence C_3^4 must have 3 cusps.

8a. The corresponding curve of genus 0 and class 4 must have 3 points of inflexion.

9. Since the class of C_3^4 is 3 it must have a bitangent or a point of inflexion. Looking at Figure 2 it is not hard to decide that l_∞ must be a bitangent. It is easily proved that I and J are the points of contact.

9a. It is evident that the tangents to the inscribed circle i.e. the isotropic lines through O cut the perpendicular to the angle bisectors at O and that therefore the point O is a double point of Γ_4^3. From the statement in 9 it then follows that the isotropic lines at O are the nodal tangents at O.

Part II. Given a Family of Wallace Lines, to Find the Corresponding Triangle

It is a well known fact that any point of the quadruple formed by the vertices ABC of a triangle and its orthocentre H may be considered to be the orthocentre of the triangle formed by the other three. Also it is known that these four triangles though having different circumcircles (of the same radius) have their ninepoints circle and the family of Wallace lines in common.

Now one may pose the question: Given the set of Wallace lines and their envelope find the triangle and its orthocentre.

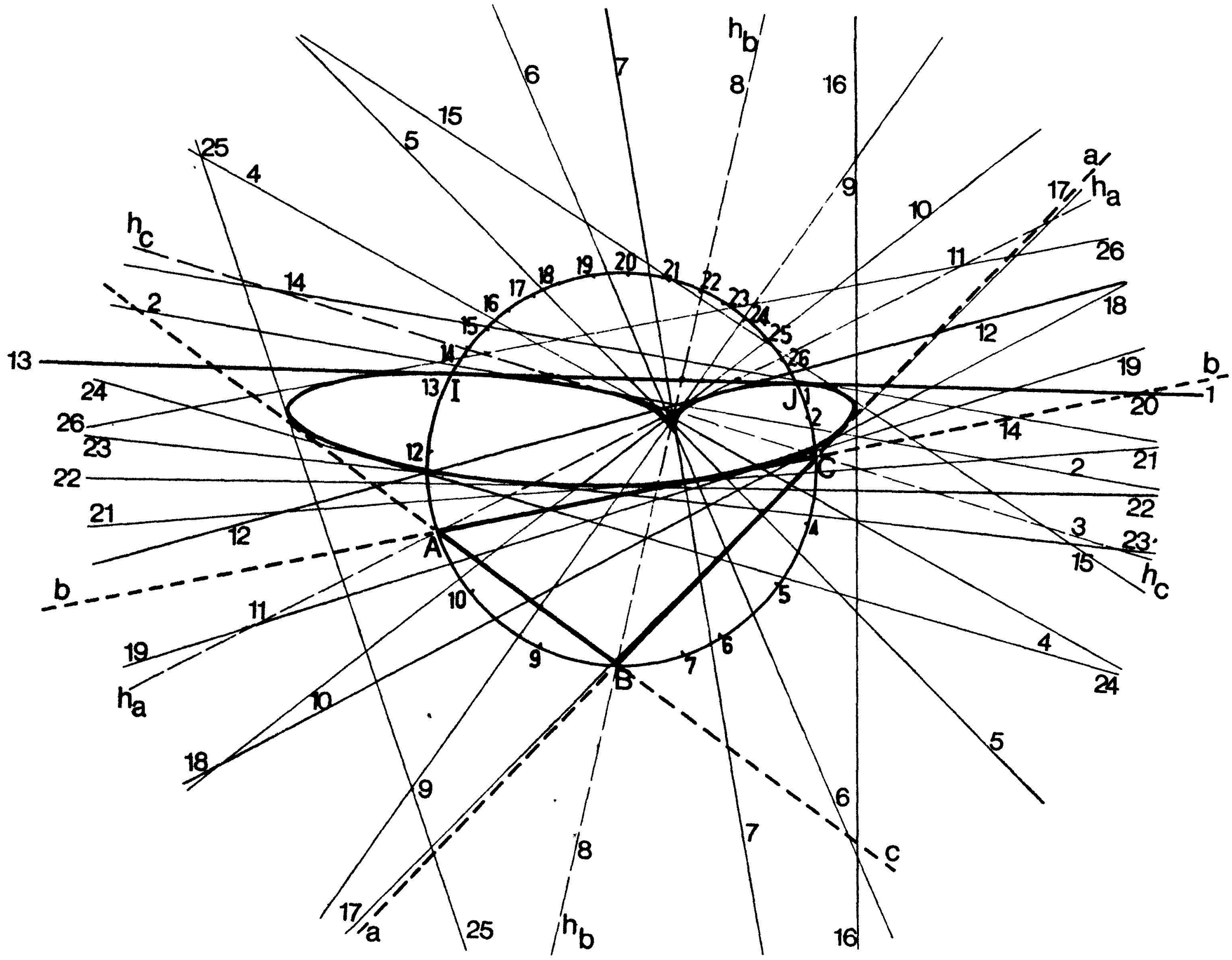

Figure 2

The answer is not difficult if one knows, that the envelope – a hypocycloid with three cusps – has the same centre as the ninepoints circle and touches this circle in three points; further that the ninepoints circle is the locus of points where two of the three Wallace lines meet at right angles.

Now to find the triangle we choose an arbitrary point A. One of the three Wallace lines through A we call h_a. This line h_a cuts the ninepoints circle in two points, in one and only one of the two there will be a Wallace line orthogonal to h_a. Call this last line a. Where a is cut by the other two lines through A lie the other two vertices B and C of the required triangle. It transpires that there are ∞^2 triangles which generate the same family of Wallace lines.

The set of quadruples of orthogonal points related to the set of Wallace lines form an involution of the fourth degree in the plane.

Consider one arbitrary Wallace line say h_a and its orthogonal line a, then each carries a second degree involution where each pair A, H on h_a is conjugated to one pair B, C on a. To realise that this is so, just let A slide along the fixed h_a, the two other Wallace lines b, c through A will change position and cut a at different pairs B, C. The position of H on h_a relative to each pair B, C may then be fixed.

The second degree involution on each Wallace line is such that the two double points of the involution coincide. The locus of these double points is the nine points circle.

To prove these two statements consider Figure 3.

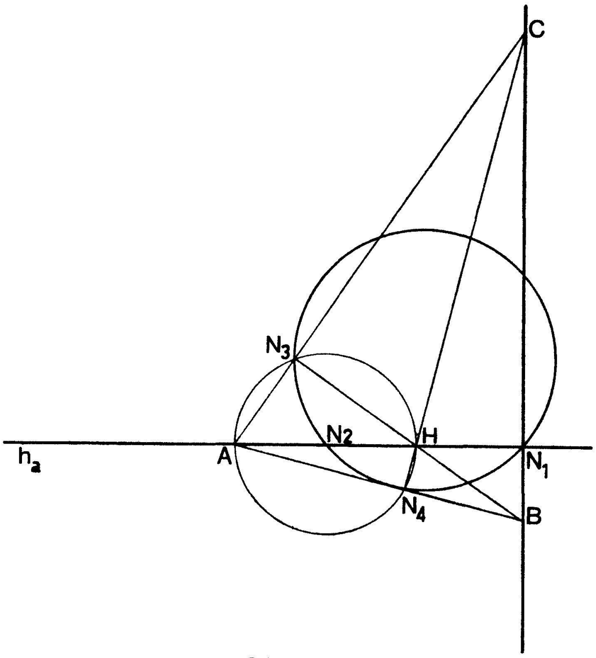

Figure 3

Assume the nine points circle and h_a and a, two Wallace lines which meet at right angles at N_1, as given. Choosing an arbitrary point A on h_a describe a circle, centre N_2, radius $N_2 A$. This circle cuts the nine points circle at N_3 and N_4 (the footpoints of the altitudes of $\triangle ABC$ from B and C) and it cuts h_a at A and H (the orthocentre of $\triangle ABC$). It is evident that as A approaches N_2 that H will approach at the same speed. Hence the involution has N_2 as coinciding double points. As A approaches N_1, B will approach N_1. Of course N_1 may be considered also as the coinciding double points of the involution on the third Wallace line passing through N_1.

Part III. The Wallace Theorem and its Dual Connected Through a Twisted Cubic

The previous theorems may be linked together by a twisted cubic.

The osculating planes of a twisted cubic envelope a ruled surface of the fourth degree. The intersection of this surface with an arbitrary plane is a fourth degree curve with three cusps. Moreover the curve is of class three.

Automatically now the question arises whether this plane curve and its tangents may be regarded as the projection of a three-cusp hypocycloid and its Wallace lines.

That this is so may be proved in the following way. Consider the coordinate-tetrahedron $O_1O_2O_3O_4$ and the twisted cubic

$$\left.\begin{aligned} x_1 &= t^2 + t + 1 = (t-b)(t-b^2), & y_1 &= 3t \\ x_2 &= b\,t^2 + b^2 t + 1 = b(t-b^2)(t-1), & y_2 &= 3t^2 \\ x_3 &= b^2 t^2 + b\,t + 1 = b^2(t-1)(t-b), & y_3 &= 3 \\ x_4 &= -t^3 + 1 = -(t-1)(t-b)(t-b^2), & y_4 &= 3(1-t^3) \end{aligned}\right\} \quad (1)$$

where $b = (-1 + i\sqrt{3})/2$ hence $1 + b + b^2 = 0$.

The equation of the osculating plane at the point with parameter t is

$$(t-1)^3 x_1 + (t-b)^3 x_2 + (t-b^2)^3 x_3 + 3 x_4 = 0\,. \tag{2}$$

Osculating planes at $t = 0$ and $1/t = 0$ are $x_1 + x_2 + x_3 = 3 x_4$ and $x_1 + x_2 + x_3 = 0$ respectively.

These planes intersect along

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_4 \qquad\qquad\quad = 0\,. \end{cases}$$

The family of lines in $x_4 = 0$ has coordinates:

$$\{(t-1)^3,\ (t-b)^3,\ (t-b^2)^3\}\,.$$

Parameter equation of the envelope is:

$$\left.\begin{aligned} \varrho\, x_1 &= t^4 + 2t^3 + 3t^2 + 2t + 1 \\ \varrho\, x_2 &= b\,t^4 + 2b^2 t^3 + 3t^2 + 2bt + b^2 \\ \varrho\, x_3 &= b^2 t^4 + 2bt^3 + 3t^2 + 2b^2 t + b \end{aligned}\right\} \tag{3}$$

The envelope touches $x_1 + x_2 + x_3 = 0$ at $\{1, b^2, b\}$ and at $\{1, b, b^2\}$.

Eliminating t from (3) we find:

$$2\,(x_1 + x_2 + x_3)\, x_1 x_2 x_3 = x_2^2 x_3^2 + x_3^2 x_1^2 + x_1^2 x_2^2 \tag{4}$$

Using the transformation

$$\left.\begin{aligned} y_1 &= x_1 + b\,x_2 + b^2 x_3 \\ y_2 &= x_1 + b^2 x_2 + b\,x_3 \\ y_3 &= x_1 + x_2 + x_3 \end{aligned}\right\} \ (5) \qquad \left.\begin{aligned} 3x_1 &= y_1 + y_2 + y_3 \\ 3x_2 &= b^2 y_1 + b\,y_2 + y_3 \\ 3x_3 &= b\,y_1 + b^2 y_2 + y_3 \end{aligned}\right\} \ (5a)$$

the equation (4) may be replaced by:

$$y_3^4 - 6\,y_3^2 y_1 y_2 + 4\,y_3 (y_1^3 + y_2^3) - 3\,y_1^2 y_2^2 = 0 \tag{6}$$

Consider a point H with x-coordinates $(h_1, h_2, h_3, 0)$. To find the osculating planes passing through H we have to solve the equation

$$h_1 (t-1)^3 + h_2 (t-b)^3 + h_3 (t-b^2)^3 = 0$$

or

$$(h_1 + h_2 + h_3)\, t^3 - 3t^2 (h_1 + b\,h_2 + b^2 h_3) + 3t\,(h_1 + b^2 h_2 + b\,h_3) - (h_1 + h_2 + h_3) = 0$$

or

$$H_3 t^3 - 3 H_1 t^2 + 3 H_2 t - H_3 = 0$$

where H_1, H_2 and H_3 are the y-coordinates of H.

Writing the last equation $H_3 (t-\alpha)(t-\beta)(t-\gamma) = 0$ where

$$\alpha\beta\gamma = 1\,, \qquad \alpha + \beta + \gamma = \frac{3 H_1}{H_3}\,, \qquad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{3 H_2}{H_3}\,,$$

considering the osculating planes at $t = -\alpha$, $t = -\beta$, $t = -\gamma$ we find that the intersection of the last two with $x_4 = 0$ is the point A with y-coordinates (A_1, A_2, A_3).

The values of A are found from:

$$A_3(-\beta^3 - 1) - 3A_1\beta^2 - 3A_2\beta = 0$$

$$A_3(-\gamma^3 - 1) - 3A_1\gamma^2 - 3A_3\gamma = 0$$

whence

$$A_1 = \frac{1}{\beta\gamma} - \beta - \gamma = \alpha - \beta - \gamma, \quad A_2 = \beta\gamma - \frac{1}{\beta} - \frac{1}{\gamma} = \frac{1}{\alpha} - \frac{1}{\beta} - \frac{1}{\gamma}, \quad A_3 = 3.$$

The third line through A is found from

$$A_3(t + \beta)(t + \gamma)(t - \xi) = 0$$

where $\beta\gamma\xi = 1$

$$-(\beta + \gamma) + \xi = \frac{3A_1}{A_3} = \alpha - \beta - \gamma$$

$$-\frac{1}{\beta} - \frac{1}{\gamma} + \frac{1}{\xi} = \frac{3A_2}{A_3} = \frac{1}{\alpha} - \frac{1}{\beta} - \frac{1}{\gamma}$$

obviously $\xi = \alpha$ satisfies these equations.

We have now six points on the twisted cubic, with parameter values $\pm\alpha$, $\pm\beta$, $\pm\gamma$. The corresponding osculating planes cut $x_4 = 0$ as in the following diagram.

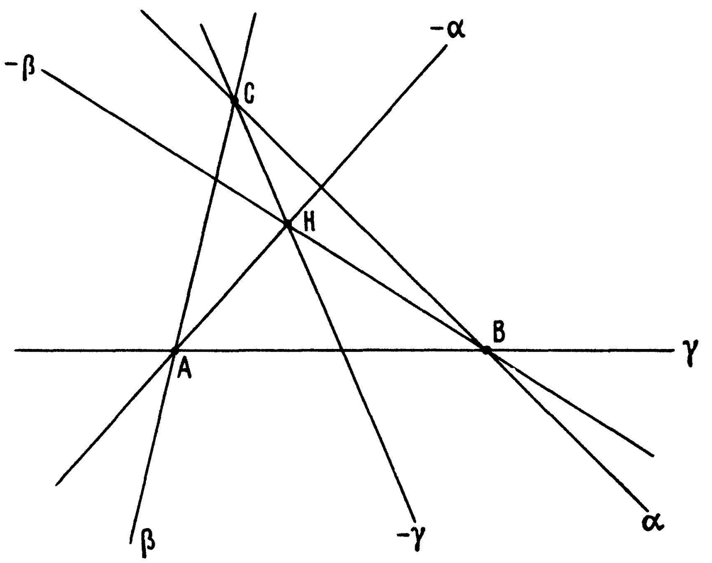

Figure 4

The equation of the locus of points where two of the quadruple A, B, C, H coincide is found from

$$\frac{\alpha + \beta + \dfrac{1}{\alpha\beta}}{\alpha - \beta - \dfrac{1}{\alpha\beta}} = \frac{\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \alpha\beta}{\dfrac{1}{\alpha} - \dfrac{1}{\beta} - \alpha\beta} = \frac{3}{3}, \qquad \beta = \frac{\pm i}{\sqrt{\alpha}}, \qquad \gamma = \frac{\mp i}{\sqrt{\alpha}}$$

whence $9y_1y_2 = y_3^2$. This conic corresponds to the nine points circle in the Euclidian case.

Briefly considering the dual case, we find that a plane through the points with parameter values $(-\alpha)$, $(-\beta)$ and $(-\gamma)$ has the equation

$$y_1 \sum \alpha\beta + y_2 \sum \alpha + y_3 (1 + \alpha\beta\gamma) - y_4 = 0 ,$$

for $(-\alpha) \cdot (-\beta) \cdot (-\gamma) = -1$ this becomes

$$y_1 \sum \alpha\beta + y_2 \sum \alpha + 2 y_3 - y_4 = 0 .$$

Obviously the four planes passing through the triplets

$$(-\alpha)(-\beta)(-\gamma) , \quad (-\alpha)(+\beta)(+\gamma) , \quad (+\alpha)(-\beta)(+\gamma) \quad \text{and} \quad (+\alpha)(+\beta)(-\gamma)$$

all pass through (0, 0, 2, 1).

Projecting the twisted cubic from (0, 0, 2, 1) onto $y_4 = 0$ produces a plane cubic Γ_4^3

$$y_1 = t = s^2 \qquad y_2 = t^2 = s$$

$$y_3 = \frac{1+t^3}{2} = \frac{s^3+1}{2} \quad \text{or} \quad y_1^3 + y_3^3 = 2 y_1 y_2 y_3 \qquad y_4 = 0$$

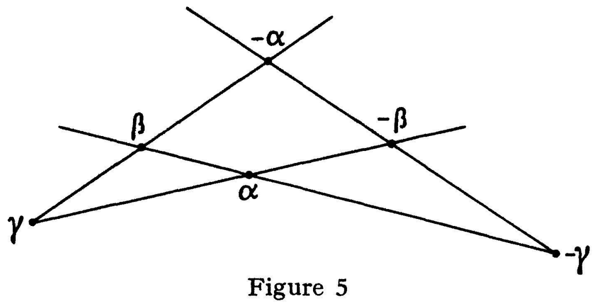

Figure 5

which has a double point at (0, 0, 1, 0) and which cuts the four planes mentioned above as in Figure 5

To find the dual to the nine points circle note that through each point on this circle pass two Wallace lines which are perpendicular to each other. So now we have to find a line that cuts Γ_4^3 in three points of which two have a double ratio -1 with the intersection of the lines i and j with that line. From previous results we may expect, that the projection of the points with parameter values α and $-\alpha$ are such a pair. This is easily verified.

The line joining the points

$$\left(\alpha, \alpha^2, \frac{1+\alpha^3}{2}, 0\right) \quad \text{and} \quad -\left(-\alpha, \alpha^2, \frac{1-\alpha^3}{2}, 0\right)$$

meets Γ_4^3 also at

$$\left(\alpha^4, \alpha^2, \frac{1+\alpha^6}{2}, 0\right).$$

Its equations are

$$\alpha^4 y_1 + y_2 - 2\alpha^2 y_3 = 0 , \quad y_4 = 0 .$$

It is tangent to the curve

$$y_1 = 1 , \quad y_2 = \alpha^4 , \quad y_3 = -\alpha^2 \ \text{or} \ y_1 y_2 = y_3^2$$

which is a conic with one focal point coinciding with the double point of Γ_4^3.

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