# Application of non-linear programming to plane geometry

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## 1. Introduction

The purpose of this paper is to show how one may obtain geometric inequalities by means of purely non-geometric methods. An advantage of this approach is that those inequalities may be viewed in a somewhat wider setting than that given by mere plane geometry. Although we intend to prove only two inequalities, we strongly feel that others may be found in a similar fashion. The method to be used is taken from the field of non-linear programming, to be more specific, we shall employ an adopted version of the Kuhn-Tucker theorem.
To illustrate our point, we have selected the following inequalities:

$$ab+bc+ca<k_1(a+b+c)^2 \quad \text{with} \quad k_1=-\frac{5}{2}+2\sqrt{2} \tag{1.1}$$

and

$$(a\beta-b\alpha)^2+(b\gamma-c\beta)^2+(c\alpha-a\gamma)^2<k_2(a+b+c)^2 \quad \text{with} \quad k_2=\frac{1}{4}\pi^2. \tag{1.2}$$

In (1.1) the quantities $a, b$ and $c$ stand for the sides of an *obtuse* triangle and in (1.2) $a, b$ and $c$ are the sides and $\alpha, \beta$ and $\gamma$ are the corresponding angles (measured in radials) of an *arbitrary* triangle.
The first inequality is proved in [3] by means of an entirely geometric argument. Note that (1.1) with constant $k_1=1/3$ holds for any non-equilateral triangle. However in that case the inequality becomes rather trivial (cf. [2], 1.1, p. 11).
The second inequality has more stature. A proof may be found in [5]. This proof uses both geometric and non-geometric methods. See also [2], 3.5, p. 38.

## 2. The Kuhn-Tucker theorem

Let $f, g_1, \ldots, g_m$ be real-valued functions defined on a subset $X$ of $\mathbf{R}^n$. Optimization problems, which can be put into the form

$$\text{Maximize } f(x), \text{ subject to } \begin{cases} g_i(x) \geqq 0 & \text{for} \quad i=1, \ldots, m_1 \\ g_i(x)=0 & \text{for} \quad i=m_1+1, \ldots, m \end{cases} \quad x \in X \tag{2.1}$$

are the subject matter of what is known as *programming*; *linear* programming when the functions $f, g_1, \ldots, g_m$ are all linear functions and *non-linear* programming otherwise.
We define the set $C$ as follows:

$$C = \{x \in X \mid g_i(x) \geqq 0 \quad \text{for} \quad i = 1, \ldots, m_1 \wedge g_i(x) = 0 \quad \text{for} \quad i = m_1 + 1, \ldots, m\} \quad (2.2)$$

and we shall always assume that this so-called *constraint set* is non-empty.
If $C$ is compact (i.e. closed and bounded) and $f$ continuous, the existence of a solution to problem (2.1) is garanteed by the following well-known theorem:

**Theorem A** (Weierstrass). *Let $C$ be a compact subset of $\mathbf{R}^n$ and suppose that the function $f: \mathbf{R}^n \rightarrow \mathbf{R}$ is continuous. Then the restriction of $f$ to $C$ attains a (global) maximum and a (global) minimum.*

Often the constraint set is unbounded. In that case it is not always easy, if at all possible, to prove the existence of a solution to (2.1). The only existence theorems known for such a situation relate to concave (or convex) programming and quadratic programming.
We suppose for the moment that a solution does exist. In order to find the maximal value of $f$ attained on $C$, the following theorem could be of some use, although in practice it is not often applied in a constructive way.

**Theorem B** (Kuhn-Tucker). *Let $f$, $g_1$, ..., $g_m$ be real-valued totally differentiable functions defined on a non-empty open subset $X$ of $\mathbf{R}^n$. Further, let $C$ be defined as in (2.2). For every $x \in C$, we define $E(x)$ to be the set of all indices $j \in \{1, \ldots, m_1\}$ for which $g_j(x) = 0$. Moreover, let $f$ attain a local maximum on $C$ in the point $\hat{x}$. Assume that at least one of the following regularity conditions is satisfied:*

R1. *All contraint functions $g_i$ are linear;*
R2. *The set of gradient vectors $\{\nabla g_i(\hat{x}) \mid i \in E(\hat{x}) \vee i \in \{m_1 + 1, \ldots, m\}\}$ is linear independent.*

*Then the following conditions* (first order *conditions or* Kuhn-Tucker *conditions) are fulfilled:*
*There exist real numbers $\lambda_1, \ldots, \lambda_m$ such that*

$$\begin{aligned} &\nabla f(\hat{x}) + \sum_{i=1}^{m} \lambda_i g_i(\hat{x}) = 0 \\ &\lambda_i g_i(\hat{x}) = 0, \qquad i = 1, \ldots, m \\ &g_i(\hat{x}) \geqq 0 \quad \text{and} \quad \lambda_i \geqq 0, \qquad i = 1, \ldots, m_1 \\ &g_i(\hat{x}) = 0, \qquad i = m_1 + 1, \ldots, m. \end{aligned} \qquad (2.3)$$

Remarks: The notation $\nabla f(\hat{x})$ stands for 'the gradient of $f$ in $\hat{x}$' i.e.
$\nabla f(\hat{x}) := (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)_{x=\hat{x}}$.
Proofs of theorem B can be found in various places e.g. [1], p. 121.

There exist a wide variety of regularity conditions (cf. [4], chapter 1, section D). We have chosen R1 and R2 merely, because they prove sufficient for the applications selected.
On reversing the relevant inequality signs and replacing the phrase 'local maximum' by 'local minimum' in theorem B, we obtain an analoguous theorem for the problem:

$$\text{Minimize } f(x), \text{ subject to } \begin{cases} g_i(x) \leqq 0 & \text{for } i=1,\dots,m_1 \\ g_i(x)=0 & \text{for } i=m_1+1,\dots,m \end{cases} \quad x \in X \tag{2.1$'$}$$

## 3. Applications to plane geometry

In this section we shall give proofs of the inequalities mentioned in the introduction.

**Lemma 1.** *The problem*

$$max\, f(x) = x_1x_2 + x_2x_3 + x_3x_1$$

*subject to*

$$\begin{array}{lll} x_1 \geqq 0, & x_2 \geqq 0, & x_3 \geqq 0 \\ x_1^2 \geqq x_2^2 + x_3^2 & & \\ x_1 + x_2 + x_3 = 1 & & \end{array}$$

*has a solution. This maximum is attained in one point only, namely* $\hat{x} = (-1+\sqrt{2},\ 1-\sqrt{2}/2,\ 1-\sqrt{2}/2)$ *and* $f(\hat{x}) = -5/2 + 2\sqrt{2}$.

Proof: Clearly $f$ is a continuous function on $\mathbf{R}^3$ and the constraint set $C$ is compact. This shows the existence of a solution $M$, attained in a point $\hat{x} = (x_1, x_2, x_3)$ say. Since $\hat{x} \in C$, it is clear that $x_1 \neq 0$. Moreover, if $x_2 = x_3 = 0$, then $M = 0$. However, $f$ is not identically zero on $C$. So $x_2$ and $x_3$ cannot vanish simultaneously. Now suppose that $x_2x_3 = 0$. Because of symmetry, we may assume that $x_2 = 0$ and $x_3 \neq 0$. Then $M = x_1x_3 \leqq 1/4$, in view of the relation $x_1 + x_3 = 1$. On the other hand, $f(5t, 4t, 3t) = 47t^2$ and $(5t, 4t, 3t) \in C$ iff $t = 1/12$. But $f(5/12,\ 4/12,\ 3/12) > 1/4$. Consequently, $x_2x_3 \neq 0$. It is now easy to check that condition R2 of theorem B is satisfied in $\hat{x}$. Hence, real numbers $\lambda_1, \lambda_2, \lambda_3, \mu, \nu$ exist such that (see (2.3)):

$$\begin{array}{llll} & & x_2 + x_3 + \lambda_1 + 2\mu x_1 + \nu = 0 & \\ & & x_1 + x_3 + \lambda_2 - 2\mu x_2 + \nu = 0 & \\ & & x_1 + x_2 + \lambda_3 - 2\mu x_3 + \nu = 0 & \\ x_1 \geqq 0, & x_2 \geqq 0, & x_3 \geqq 0 & \lambda_1 x_1 = \lambda_2 x_2 = \lambda_3 x_3 = 0 \\ x_1^2 \geqq x_2^2 + x_3^2 & & & \mu(x_1^2 - x_2^2 - x_3^2) = 0 \\ x_1 + x_2 + x_3 = 1 & & & \nu(x_1 + x_2 + x_3 - 1) = 0 \\ \lambda_1 \geqq 0, & \lambda_2 \geqq 0, & \lambda_3 \geqq 0, & \mu \geqq 0. \end{array}$$

Since $x_1x_2x_3 \neq 0$, it follows that $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

From the first three equations we obtain by addition

$$\begin{aligned}0&=2(x_1+x_2+x_3)+\lambda_1+\lambda_2+\lambda_3+2\mu(x_1-x_2-x_3)+3\nu\\&=2+2\mu(x_1-x_2-x_3)+3\nu.\end{aligned}$$

If $\mu=0$, then $\nu=-2/3$ and thus $1-x_1=1-x_2=1-x_3=2/3$. Hence $x_1=x_2=x_3=1/3$, but this contradicts $x_1^2\geqq x_2^2+x_3^2$. Thus $\mu>0$ and consequently $x_1^2=x_2^2+x_3^2$. From

$$\nu+1=x_1(1-2\mu)=x_2(1+2\mu)=x_3(1+2\mu)$$

it follows that $x_2=x_3$ and hence $x_1^2=2x_2^2$ which gives $x_1=x_2\sqrt{2}$. Then $x_1+x_2+x_3=1$ shows that $x_l=-1+\sqrt{2}$ and $x_2=x_3=1-\sqrt{2}/2$. We also find $\mu=3/2-\sqrt{2}$ and $\nu=5-4\sqrt{2}$.

From this lemma, the following theorem can be easily deduced.

**Theorem 1.** *Let $a, b$ and $c$ be the sides of an obtuse triangle. Then inequality* (1.1) *holds and the constant $k_1$ is best possible.*

Proof: Put $x_1=a/(a+b+c)$, $x_2=b/(a+b+c)$ and $x_3=c/(a+b+c)$. The quantities $x_1, x_2, x_3$ satisfy $x_1>0$, $x_2>0$, $x_3>0$, $x_1+x_2+x_3=1$ and $x_1^2>x_2^2+x_3^2$ if we assume, without loss of generality, that $a=\max(a,b,c)$.

Lemma 1 shows that equality can only be reached in a right isosceles triangle with $2b=2c=a\sqrt{2}$. That $k_1$ is best possible also follows from the observation that for each sufficiently small positive number $\delta$, the triangle with sides $a=-1+\sqrt{2}+\delta, 2b=2c=2-\sqrt{2}-\delta$ is obtuse.

Inequality (1.2) is somewhat harder to prove. We need the following lemma.

**Lemma 2.** *The problem*

$$max\, f(x;y)=(x_1y_2-x_2y_1)^2+(x_2y_3-x_3y_2)^2+(x_3y_1-x_1y_3)^2$$

*subject to*

$$\begin{aligned}x_1+x_2+x_3&=1, & y_1+y_2+y_3&=1\\-x_1+1/2&\geqq 0 & &\\x_1-x_2&\geqq 0 & y_1-y_2&\geqq 0\\x_2-x_3&\geqq 0 & y_2-y_3&\geqq 0\\x_3&\geqq 0 & y_3&\geqq 0\end{aligned}$$

*is solvable. The maximum $M=1/4$ is attained at $\hat{x}=(1/2,\ 1/2,\ 0;\ 1,\ 0,\ 0)$ and at no other point of $C$.*

Proof: The function $f$ is continuous on $\mathbf{R}^6$ and the constraint set $C$ is compact. Let $f$ attain its maximum $M$ on $C$ in the point $\hat{x}=(x_1,x_2,x_3;y_1,y_2,y_3)$. Since all constraint functions are linear, the regularity condition R1. of theorem B is fulfilled. Hence, there exist real numbers $a_0$, $a_1$, $a_2$, $a_3$, $\beta_1$, $\beta_2$, $\beta_3$, $\lambda$, $\mu$ such that (see (2.3) of theorem B):

$$\frac{\partial f}{\partial x_1}-a_0+a_1+\lambda=0, \qquad \frac{\partial f}{\partial y_1}+\beta_1+\mu=0$$

$$\frac{\partial f}{\partial x_2}-a_1+a_2+\lambda=0 \qquad \frac{\partial f}{\partial y_2}-\beta_1+\beta_2+\mu=0$$

$$\frac{\partial f}{\partial x_3}-a_2+a_3+\lambda=0 \qquad \frac{\partial f}{\partial y_3}-\beta_2+\beta_3+\mu=0$$

$$\begin{array}{llll} -x_1+1/2\geqq 0, & a_0(-x_1+1/2)=0 & & \\ x_1-x_2\geqq 0 & a_1(x_1-x_2)=0, & y_1-y_2\geqq 0, & \beta_1(y_1-y_2)=0 \\ x_2-x_3\geqq 0 & a_2(x_2-x_3)=0 & y_2-y_3\geqq 0 & \beta_2(y_2-y_3)=0 \\ x_3\geqq 0 & a_3x_3=0 & y_3\geqq 0 & \beta_3y_3=0 \\ x_1+x_2+x_3=1 & & y_1+y_2+y_3=1 & \end{array}$$

$$a_0\geqq 0,\quad a_1\geqq 0,\quad a_2\geqq 0,\quad a_3\geqq 0,\quad \beta_1\geqq 0,\quad \beta_2\geqq 0,\quad \beta_3\geqq 0.$$

First of all we note that $f(1/2, 1/2, 0; 1, 0, 0)=1/4$. Hence $M=\max f\geqq 1/4$.
Since $3y_3\leqq y_1+y_2+y_3=1$ and $0\leqq x_2\leqq x_1\leqq 1/2$, we have $\beta_2-\beta_3-\mu=\partial f/\partial y_3$ $=2y_3(x_1^2+x_2^2)-2x_3(x_1y_1+x_2y_2)\leqq 1/3$.

Moreover, as a function of $y_1, y_2, y_3$ alone, the function $f$ is homogeneous of degree 2. Hence, by Euler's theorem

$$2f=y_1\frac{\partial f}{\partial y_1}+y_2\frac{\partial f}{\partial y_2}+y_3\frac{\partial f}{\partial y_3}=-\mu.$$

Combining these two results, we obtain

$$2f+\beta_2-\beta_3\leqq\frac{1}{3}.$$

Now, if $\beta_3=0$ then $\beta_2\geqq 0$ implies that $f\leqq 1/6$. This means that we may assume that $\beta_3>0$. But then $y_3=0$.
As a function of $x_1, x_2, x_3$ alone, the function $f$ is also homogeneous of degree 2. Hence, as before,

$$2f=x_1\frac{\partial f}{\partial x_1}+x_2\frac{\partial f}{\partial x_2}+x_3\frac{\partial f}{\partial x_3}=-\lambda+\frac{a_0}{2}.$$

Further, $\partial f/\partial x_1+\partial f/\partial x_2=a_0-a_2-2\lambda=-a_2+4f$ and also $\partial f/\partial x_1+\partial f/\partial x_2$ $=2x_1(y_1-y_2)^2-2y_1(x_1-x_2)(y_1-y_2)\leqq 1$, since $y_3=0$. Thus

$$-a_2+4f\leqq 1.$$

Suppose now that $a_2=0$. Because we are only interested in values of $f \geqq 1/4$, it follows from the above that

$$1 \leqq 4f = 2x_1(y_1-y_2)^2 - 2y_1(x_1-x_2)(y_1-y_2) \leqq 1$$

and this means that

$$2x_1(y_1-y_2)^2 = 1 \quad \text{and} \quad 2y_1(x_1-x_2)(y_1-y_2) = 0.$$

This is only possible when $x_1=x_2=1/2$ and $y_1=1, y_2=0$. Consequently, $x_3=0$. After some calculation we find that $0 \leqq a_0 \leqq 1$, $a_1=(1+a_0)/2$, $(a_2=0)$, $a_3=(1-a_0)/2$, $\beta_1=0$, $\beta_2=1$ and $\beta_3=3/2$. Hence the first order conditions are satisfied in the point $(1/2, 1/2, 0; 1, 0, 0)$.
We continue by assuming that $\hat{x} \neq (1/2, 1/2, 0; 1, 0, 0)$. Then clearly $a_2>0$ and $x_2=x_3$. Now $2f+a_0/2-a_1=\partial f/\partial x_1=2y_2(x_1y_2-x_2y_1)=2x_1y_2(y_2-y_1)+2y_1y_2(x_1-x_2) \leqq y_1y_2 \leqq 1/4$, because $y_1+y_2=1$ (recall that $y_3=0$). Hence,

$$2f-a_1 \leqq \frac{1}{4},$$

in view of $a_0 \geqq 0$. From $a_1=0$, it follows that $f \leqq 1/8$. Hence suppose that $a_1>0$. Then $x_1=x_2$. Also $x_2=x_3$ and thus $x_1=x_2=x_3=1/3$. We have

$$4f-\beta_2=\frac{\partial f}{\partial y_1}+\frac{\partial f}{\partial y_2}=\frac{2}{9}(y_1+y_2)=\frac{2}{9}.$$

If $\beta_2=0$, then $f=1/18$. And if $\beta_2>0$, then $y_2=y_3=0$ and thus $y_1=1$. This implies that $2f=\partial f/\partial y_1=4/9$, since $\beta_1=0\,(y_1 \neq y_2)$. But then $f=2/9<1/4$.
This proves the lemma.

**Theorem 2.** *Let $a$, $b$, $c$ be the sides and $\alpha$, $\beta$, $\gamma$ the corresponding angles of a triangle. Then inequality (1.2) holds. Moreover, the constant $k_2$ is best possible.*

Proof: Put $x_1=a/(a+b+c)$, $x_2=b/(a+b+c)$, $x_3=c/(a+b+c)$ and assume that $a \geqq b \geqq c$. Further, put $y_1=\alpha/\pi$, $y_2=\beta/\pi$ and $y_3=\gamma/\pi$. Then $x_1+x_2+x_3=y_1+y_2+y_3=1$. Since $b+c>a$, we have also $x_2+x_3>x_1$. This shows that $x_1<1/2$. In view of $a \geqq b \geqq c$, we have $\alpha \geqq \beta \geqq \gamma$ and consequently $x_1 \geqq x_2 \geqq x_3>0$ and $y_1 \geqq y_2 \geqq y_3>0$. That $k_2$ is best possible may be seen as follows (in fact the proof of lemma 2 already gives evidence to that effect):
Let $\delta>0$. Put $y_1=\alpha/\pi=1-(\delta+\delta^2)/\pi$, $y_2=\beta/\pi=\delta/\pi$, $y_3=\gamma/\pi=\delta^2/\pi$ and $x_1=a=\sin(\delta+\delta^2)/2\delta$, $x_2=b=\sin\delta/2\delta$, $x_3=c=\sin\delta^2/2\delta$. Now let $\delta$ tend to zero.

## 4. Postscript

The most difficult part of the foregoing method in order to obtain geometric inequalities, lies in the choice of the constraint set. The relations between the elements of a triangle are often given in terms of circle functions. These functions, when appearing in the constraint functions, greatly complicate the determination of points satisfying the first order conditions.

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# Distance theorems in geometry

## 1. Introduction

The purpose of this note is to give a method for proving 'distance theorems' in elementary plane geometry. As an application we give an easy proof of the Feuerbach theorem and we solve an old problem of A. H. Stone [3] problem E585.

Let $(T)$ be any triangle $A_1A_2A_3$ with vertices numbered in counter clockwise order. Denote the interior angle at $A_i$ by $\alpha_i (i=1,2,3)$, and the length of the opposite side by $a_i$. We use the notation $P(x_1, x_2, x_3)$ or $(x_i)$ to indicate that the distances of $P$ from the sides of $(T)$ are proportional to $x_1, x_2, x_3$ with the convention that $x_i$ is positive if $P$ and $A_i$ are on the same side of $a_i$ and negative otherwise. We shall also use capital letters to denote complex numbers; thus, for example, $(1/3)$ $(A_1+A_2+A_3)$ is the centroid of $(T)$.

Our method is based on the following elementary lemma.

**Lemma.** *Let $M$ be a point in the plane of $(T)$ satisfying*

$$\sum_{i=1}^{3} m_i \overline{MA}_i^2 = k, \tag{1}$$

*where the $m_i$'s are real numbers satisfying $s_3 = m_1+m_2+m_3 \neq 0$, and $k$ is a constant satisfying*