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Let $V(K)$ denote the volume of a convex body K in the Euclidean d-space E^d, and let $G(K)$ and $G^0(K)$ be defined by $G(K)=\text{card}(K\cap Z^d)$ and $G^0(K)=\text{card}(\text{int } K\cap Z^d)$, where Z^d denotes the set of all the lattice points in E^d, i.e., points having only integral coordinates.
The following theorems of Minkowski are well known.

Theorem A (see [2], p. 76). *If K is a centrally symmetric convex body in E^d and $G^0(K)=1$, then $V(K)\leq 2^d$.*

Theorem B (see [2], p. 79 and 96). *If K is a centrally symmetric convex body in E^d and $G^0(K)=1$, then $G(K)\leq 3^d$; if $G^0(K)=1$ and $V(K)=2^d$, then K is a convex polytope and it has at most $2^{d+1}-2$ facets.*

There are no analogues of theorems A and B for convex bodies which are not centrally symmetric. The purpose of this paper is to give some information on $V(P)$ and $G(P)$ in the case where P is a lattice polytope (i.e., P has all of its vertices in Z^d) satisfying $G^0(P)=n, n\geq 1$. We will see that V nad G behave very different from the centrally symmetric case.
Let $g(d,n)$ and $v(d,n)$ be defined for $n\geq 1$ and $d\geq 2$ as follows: $g(d,n)=\sup\{G(P)\mid P\subset E^d, G^0(P)=n\}$ and $v(d,n)=\sup\{V(P)\mid P\subset E^d, G^0(P)=n\}$.
Scott, [3] proved that $g(2,1)=10$ and that $g(2,n)=3n+6$ for all $n\geq 2$. Our main result is the following

Theorem. *For all $d, d\geq 4$, and for all $n, n\geq 1$,*

$$v(d,n)\geq \frac{n+1}{d!}\,2^{2^{d-a}} \quad \textit{and} \quad g(d,n)\geq \frac{n+1}{6(d-2)!}\,2^{2^{d-a}},$$

where $a=0.5856\ldots$; $v(3,n)\geq 6(n+1)$, $g(3,n)\geq 16n+23$, $v(4,1)\geq 147$ and $g(4,1)\geq 680$.

We need the following

Lemma. *If $(a_n)_{n=1}^{\infty}$ is defined by $a_1=2$ and $a_d=\prod_{i=1}^{d-1} a_i+1$ for $d\geq 2$, then*

(i) $a_1=2, a_2=3, a_3=7, a_4=43$ and $a_5=1807$;

(ii) for all $d, d\geq 1$, $1-\sum_{i=1}^{d}\frac{1}{a_i}=\left(\prod_{i=1}^{d} a_i\right)^{-1}$;

(iii) for all $d, d\geq 4$, $\prod_{i=1}^{d} a_i\geq 2^{2^{d-a}}$ where $a=0.5856\ldots$

Proof of the lemma: (i) is trivial, (ii) is true for $d=1$, and if it is true for d, then by induction

$$1-\sum_{i=1}^{d+1}\frac{1}{a_i}=\left(\prod_{i=1}^{d}a_i\right)^{-1}-\frac{1}{a_{d+1}}=\left(\prod_{i=1}^{d+1}a_i\right)^{-1}.$$

Let the real a be defined by $2^{2^{d-a}}=a_1a_2a_3a_4=1806$, with $d=4$; thus $a=0.5856\ldots$ Suppose (iii) holds for d, then

$$\prod_{i=1}^{d+1}a_i>\left(\prod_{i=1}^{d}a_i\right)^2\geq 2^{2\cdot 2^{d-a}}=2^{2^{d+1-a}},$$

hence it holds for $d+1$.

Proof of the theorem: Let $k_i=a_i$ for $i=1,\ldots,d-1, k_d=2(a_d-1)$, and let S_1^d be the d-simplex defined by

$$S_1^d=\left\{(x_1,\ldots,x_d)\in E^d \mid x_i\geq 0,\quad \sum_{i=1}^{d}\frac{x_i}{k_i}\leq 1\right\}.$$

By (ii) we have

$$\sum_{i=1}^{d}\frac{1}{k_i}<1 \quad\text{and}\quad \frac{1}{k_d}+\sum_{i=1}^{d}\frac{1}{k_i}=1,$$

hence $(1,\ldots,1)\in\operatorname{int}S_1^d$ and $(1,\ldots,1,2)\notin\operatorname{int}S_1^d$; $k_1<k_2<\cdots<k_d$ imply that $G^0(S_1^d)=1$. For general $n, n\geq 1$, replace $k_d=2(a_d-1)$ by $k_d=(n+1)(a_d-1)$, and obtain the d-simplex S_n^d. It follows that the only lattice points of $\operatorname{int}S_n^d$ are $(1,\ldots,1,j)$ for $1\leq j\leq n$, hence $G^0(S_n^d)=n$.
As for the volume of S_n^d, $V(S_n^d)=(1/d!)\prod_{i=1}^{d}k_i=((n+1)/d!)\,(a_d-1)^2$ and by (iii):

$$a_d-1\geq 2^{2^{d-1-a}},\qquad\text{therefore}\quad v(d,n)\geq\frac{n+1}{d!}\,2^{2^{d-a}}.$$

To compute $G(S_d^n)$ let $d\geq 3$ and $S=S_n^d\cap\{x_1=0\}\cap\{x_2=0\}$.
It is easy to see that for each $t\in\operatorname{aff}S$
$G(S+t)\leq G(S)$ and equality iff x is a lattice point.
If C_{d-2} denotes the unit cube in $\operatorname{aff}S$ and V_{d-2} the volume in $\operatorname{aff}S$, we have

$$V_{d-2}(S)=\int_{C_{d-2}}G(S+t)\,dt<G(S)<G(S_n^d).$$

With $6\,V_{d-2}(S)=d(d-1)\,V(S_n^d)$ we have

$$G(S_n^d)\leq\frac{d(d-1)}{6}\,V(S_n^d)=\frac{n+1}{6(d-2)!}\,2^{2^{d-a}}.$$

The case $d=3$: Clearly: $V(S_n^3)=6(n+1)$.
To compute $G(S_n^3)$ we count the integers $x_3 \geq 0$ satisfying

$$3(n+1)x_1+2(n+1)x_2+x_3 \leq 6(n+1)$$

for $(x_1,x_2)=(0,0), (0,1), (0,2), (1,0), (1,1)$ and $(2,0)$.
Easy computation gives $G(S_n^3)=16n+23$.
The case $d=4$: $V(S_1^4)=147$.
To compute $G(S_1^4)$ we count the pairs (x_3,x_4) of integers $x_3 \geq 0, x_4 \geq 0$ satisfying

$$42x_1+28x_2+7x_3+x_4 \leq 84$$

for $(x_1,x_2)=(0,0), (0,1), (0,2), (0,3), (1,0), (1,1), (2,0)$.
Counting yields $G(S_1^4)=680$.
This completes the proof of the theorem.
We raise the following *conjecture*:

$$v(d,n)<\infty \quad \text{for all} \quad d \geq 3 \quad \text{and} \quad n \geq 1 .$$

We remark that the conjecture implies $g(d,n)<\infty$ by Blichfeldt ([1], p.55): $G(P) \leq d!\,V(P)+d$ for nondegenerate lattice polytopes (compare [4]).
Similar problems may be asked for the number of i-dimensional faces of convex lattice polytopes P satisfying $G^0(P)=n$, where $0 \leq i \leq d-1$.

J. Zaks, University of Haifa and C.R.M.A., University of Montreal,
M. A. Perles, Hebrew University, Jerusalem,
J. M. Wills, University of Siegen, West Germany

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Kleine Mitteilungen

A note on the successive remainders of the exponential series

1. For real $x \neq 0$, we have by Taylor's theorem

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}\phi_n(x), \tag{1.1}$$

where

$$\phi_n(x) = e^{x\theta_n(x)}, \qquad 0 < \theta_n(x) < 1 . \tag{1.2}$$