

**Zeitschrift:** Elemente der Mathematik  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 39 (1984)  
**Heft:** 5

**Artikel:** An Egyptian algorithm for polynomials  
**Autor:** Dobbs, David E. / McConnel, Robert M.  
**DOI:** <https://doi.org/10.5169/seals-38024>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 06.07.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

# An Egyptian algorithm for polynomials

## 1. Introduction and notation

The Egyptian, or Ahmes, algorithm asserts that any positive proper fraction is a sum of finitely many so-called unit, or Egyptian, fractions. In other words, given positive integers  $a < b$ , there exist positive integers  $k_1 < \cdots < k_m$  such that  $a/b = 1/k_1 + \cdots + 1/k_m$ . A proof that any such  $a/b$  admits such a sum representation appears in [2], p. 261, and an algorithm producing such a representation is given in [1], Exercise 4(a), p. 133. This note presents an analogous result in a kindred context, that of the polynomial ring  $K[X]$ , where  $K$  is a field. It will be no more difficult to consider, more generally,  $R[X]$  where  $R$  is a domain with quotient field  $K$ .

## 2. The algorithm

It will be convenient to introduce the following terminology. A rational function  $w \in K(X)$  is said to be *R-proper* in case  $w = f/g$  for some nonzero  $f, g \in R[X]$  such that  $\deg(f) < \deg(g)$ . A natural family of examples of *R-proper* rational functions is provided in characteristic 0 by the logarithmic derivatives  $g'/g$  arising from arbitrary  $g \in K[X]$  of degree at least 1. By clearing denominators, one sees easily that  $w$  is *R-proper* if and only if  $w$  is *K-proper*. Moreover, the above definition is easily shown to be independent of the choice of  $f$  and  $g$ , by using the fact that  $K[X]$  is a unique factorization domain. Such assertions, and indeed most of the calculations in this note, may be verified via the valuation properties of  $\deg$  for polynomials over a domain. To wit,  $\deg(g_1 g_2) = \deg(g_1) + \deg(g_2)$ ; and  $\deg(g_1 + g_2) \leq \max(\deg(g_1), \deg(g_2))$ , with equality holding if  $\deg(g_1) \neq \deg(g_2)$ . A particularly useful application of these properties asserts that any nonzero sum of *R-proper* rational functions is itself *R-proper*, the point being that  $\deg(f_i) < \deg(g_i)$  for  $i = 1, 2$  implies  $\deg(f_1 g_2 + g_1 f_2) < \deg(g_1 g_2)$ . Observe that the analogous assertion for positive proper fractions is false: for instance,  $1/2 + 1/3 + 1/4 = 13/12 > 1$ .

We shall say that an *R-reciprocal representation* of a nonzero rational function  $w \in K(X)$  (of length  $m$ ) is an expression

$$w = r_1/h_1 + \cdots + r_m/h_m$$

with each  $h_i \in R[X]$ ,  $0 \neq r_i \in R$ ,  $0 < \deg(h_1) < \deg(h_2) < \cdots < \deg(h_m)$ . As any such summand  $r_i/h_i$  is *R-proper*, the result noted above implies that if  $w$  admits an *R-reciprocal* representation, then  $w$  is *R-proper*. (The analogue for  $\mathbb{Z}$  is false: vide 13/12.) Theorem (a) will establish the converse, but first we need

**Lemma.** *Let  $f, g$  be nonzero elements of  $R[X]$  such that  $\deg(f) < \deg(g)$ . If  $r_1/h_1 + \cdots + r_m/h_m$  is an *R-reciprocal* representation of  $f/g$ , then  $\deg(h_1) = \deg(g) - \deg(f)$ .*

**Proof:** Multiply both sides of  $f/g = \sum r_i/h_i$  by  $g \prod h_i$ . The result is  $f \prod h_i = g \sum_i r_i \prod_{j \neq i} h_j$ . Equating degrees, we have

$$\deg(f) + \sum \deg(h_i) = \deg(g r_1 \prod_{j \neq 1} h_j) = \deg(g) + 0 + \sum_{j \neq 1} \deg(h_j),$$

from which the conclusion is immediate.

**Theorem.** Let  $w$  be a nonzero element of  $K(X)$ . Express  $w$  as  $f/g$ , where  $f$  and  $g$  are in  $R[X]$ , and  $\deg(f) = n$ . Then:

- (a)  $w$  is  $R$ -proper if and only if there exists an  $R$ -reciprocal representation of  $w$ .
- (b) If  $w$  is  $R$ -proper, then one can algorithmically produce an  $R$ -reciprocal representation of  $w$  of length at most  $n + 1$ .

**Proof:** The 'if' half of (a) follows from the above comments. Conversely, let  $w$  be  $R$ -proper; that is,  $\deg(f) < \deg(g)$ . It is enough to show that  $w$  admits a  $K$ -reciprocal representation of the form  $\sum 1/h_i$ , of length  $m \leq n + 1$ . For then, by clearing denominators,  $1/h_i = r_i/H_i$  for suitable  $r_i \in R$  and  $H_i \in R[X]$  (with  $\deg(H_i) = \deg(h_i)$ ), and  $w$  would have an  $R$ -reciprocal representation,  $\sum r_i/H_i$ , of the same length  $m$ .

We shall show by strong induction on  $n$  that  $w$  has a  $K$ -reciprocal representation of the above form with length at most  $n + 1$ . If  $n = 0$ , this is clear, for  $f$  is then a constant and  $w = 1/f^{-1}g$ . For the induction step, use the division algorithm in  $K[X]$  to write  $g = h_1 f + r$ , for uniquely determined  $h_1$  and  $r$  in  $K[X]$  such that  $\deg(r) < n$ . Note that  $\deg(h_1) = \deg(g) - \deg(f) \geq 1$ . Moreover,

$$w = \frac{f}{g} = \frac{1}{h_1} + \frac{-r}{g h_1}. \quad (1)$$

Without loss of generality,  $r \neq 0$ . We claim that  $v = -r/g h_1$  is  $K$ -proper. Indeed,  $\deg(-r) = \deg(r) < n < \deg(g) < \deg(g) + \deg(h_1) = \deg(g h_1)$ , proving the claim. Note that we have also shown

$$\deg(r) \leq \deg(g) - 2. \quad (2)$$

Now, since  $\deg(r) < \deg(f)$ , the induction hypothesis supplies  $\sum_{i=2}^m 1/h_i$ , a  $K$ -reciprocal representation of  $v$ , with  $m - 1 \leq \deg(r) + 1 \leq n$ . By (1),  $w = 1/h_1 + \cdots + 1/h_m$ , and so it suffices to prove that  $\deg(h_1) < \deg(h_2)$ . However, the lemma gives  $\deg(h_2) = \deg(g h_1) - \deg(-r)$ , which equals  $\deg(h_1) + \deg(g) - \deg(r)$ . By (2),  $\deg(h_2) \geq \deg(h_1) + 2$ , and the proof is complete.

As noted in [2], p.261, the divergence of the harmonic series permits arbitrary positive rationals to be realized as sums of unit fractions. Essentially because  $\deg$  is non-Archimedean, the next result is the best-possible analogue for rational functions.

**Corollary.** Each element of  $K(X)$  can be expressed as a sum of the form  $h_0/r_0 + r_1/h_1 + \cdots + r_m/h_m$ , where  $r_i \in R$  for each  $i$  and  $r_0 \neq 0$ ,  $h_i \in R[X] \setminus \{0\}$  for each  $i \geq 1$  and  $h_0 \in R[X]$ , and  $\deg(h_1) < \deg(h_2) < \cdots < \deg(h_m)$ .

**Proof:** If  $u \in K(X)$ , write  $u = F/g$  for suitable  $F, g$  in  $K[X]$  with  $g \neq 0$ . By the division algorithm in  $K[X]$ ,  $F = qg + f$  for suitable  $q, f$  in  $K[X]$  with  $\deg(f) < \deg(g)$ . Thus  $u = q + f/g$ . By clearing denominators, write  $q$  as  $h_0/r_0$  for  $h_0, r_0$  as asserted. Finally, express  $f/g$  as  $r_1/h_1 + \cdots + r_m/h_m$ , either with an  $R$ -reciprocal representation of  $f/g$  (if  $f \neq 0$ ) or trivially with vanishing numerators (if  $f = 0$ ).

**Remark:** (a) If  $R = K$  then each  $R$ -proper rational function has a 'unit'  $R$ -reciprocal representation, that is, one in which each numerator  $r_i = 1$ . The converse also holds. Indeed, if  $s$  is a nonzero element of  $R$  and  $s/X$  admits an  $R$ -reciprocal representation of the type  $\sum h_i^{-1}$ , clearing denominators leads to

$$s \prod_{i=1}^m h_i = X \sum_i \left( \prod_{j \neq i} h_j \right).$$

Then, if  $a_i$  denotes the leading coefficient of  $h_i$ , equating the leading coefficients of the left- and right-sides of the display yields  $s a_1 a_2 \cdots a_m = a_2 \cdots a_m$ . By cancellation,  $s a_1 = 1$ ; that is,  $s^{-1} \in R$ .

(b) It seems natural to consider power series analogues of the above theorem. Here, we record only the simple fact that  $K[[X]]$  admits such an analogue with  $m = 1$ . More generally, let  $V$  be the valuation ring (domain) of a (discrete rank one) valuation  $v$  with value group  $\mathbf{Z}$  and local uniformizing parameter  $\pi$ . Then any 'proper' element,  $f/g$  with  $f$  and  $g$  in  $V$  and  $v(f) < v(g)$ , can be expressed as  $1/\pi^{v(g)-v(f)} u$  for an appropriate unit  $u$  of  $V$ .

(c) Unit reciprocal representations need not be unique. To see this in classical context of the Egyptian algorithm, observe that the proper fraction  $7/12$  is both  $1/3 + 1/4$  and  $1/2 + 1/12$ . An analogue for rational functions (for simplicity, with  $R = \mathbf{Q}$ ) is easily illustrated:

$$(X^2 + X + 1)/X^3 = 1/X + 1/X^2 + 1/X^3 = 1/(X-1) + 1/X^3(-X+1).$$

It is a bit harder to illustrate nonuniqueness with distinct  $\mathbf{Q}$ -reciprocal representations sharing a common initial term and having the same length. For instance, one has  $\mathbf{Q}$ -reciprocal representations

$$\begin{aligned} \frac{X^5 + X^4 + X^3 + X^2 + X - 1}{X^6 + X^3 - X^2} &= \frac{1}{X} + \frac{1}{X^2} + \frac{1}{X^3 - X} + \frac{1}{p_6} + \frac{1}{p_{12}} \\ &= \frac{1}{X} + \frac{1}{X^2 - X} + \frac{1}{-X^4 + 2X^2} + \frac{1}{p_8} + \frac{1}{p_{16}} \end{aligned} \quad (3)$$

where  $p_t$  denotes a suitable polynomial of degree  $t$ .

To indicate the genesis of (3), we first reveal a general way to synthesize a rational function admitting distinct  $K$ -reciprocal representations having a common initial term. Consider two distinct polynomials,  $f$  and  $h_1$  of the same degree  $n \geq 2$  and with the same term of highest degree; let  $r$  be a polynomial of degree at most  $n-1$  and set  $g = h_1 f + r$ . We know from the proof of the theorem that  $f/g$  has a  $K$ -reciprocal representation with initial term  $1/h_1$ . Does it also have one whose initial term is  $1/f$ ? By the lemma and theorem, it will since  $\deg(f) + \deg(g) - \deg(f^2 - g) > n$ , the point being

that  $f^2$  and  $g$  have the same leading term. One thereby obtains distinct  $K$ -reciprocal representations,  $1/h_1 + \dots = 1/f + \dots$ ; adding  $1/X$  to (the left of) each produces distinct such representations with the same initial term.

The two representations given by the above recipe may, however, have unequal lengths. Consider, for instance, the choices  $f = X^2 + 1$ ,  $h_1 = X^2 - 1$ ,  $r = X$ . The recipe produces

$$\begin{aligned} \frac{X^4 + X^3 + 2X - 1}{X^5 + X^2 - X} &= \frac{1}{X} + \frac{1}{X^2 - 1} + \frac{1}{-X^5 + X^3 - X^2 + X + 1} + \frac{1}{p_{11}} \\ &= \frac{1}{X} + \frac{1}{X^2 + 1} + \text{sum of three additional terms.} \end{aligned} \quad (4)$$

Let  $w$  denote the rational function in (4), we compute via the first equation in (4) that

$$\begin{aligned} v &= 1/X + w/X = \frac{X^5 + X^4 + X^3 + X^2 + X - 1}{X^6 + X^3 - X^2} \\ &= \frac{1}{X} + \frac{1}{X^2} + \frac{1}{X^3 - X} + \frac{1}{-X^6 + X^4 - X^3 + X^2 + X} + \frac{1}{p_{12}}, \end{aligned}$$

which explains the origin of the first equation in (3). How does one obtain the second equation in (3)? Simply apply the Theorem's algorithm to  $w$ , multiply the result through termwise by  $1/X$ , and then add  $1/X$  to (the left of) the ensuing expression. Finally, it is amusing to note that an application of the Theorem's algorithm directly to  $v$  produces yet another distinct  $\mathbf{Q}$ -reciprocal representation of  $v$ .

David E. Dobbs and Robert M. McConnel  
University of Tennessee, Knoxville (USA)

## REFERENCES

- 1 W.J. LeVeque: Fundamentals of number theory. Addison-Wesley, Reading 1977.
- 2 I. Niven and H.S. Zuckerman: An introduction to the theory of numbers, 3rd ed. Wiley, New York 1972.

© 1984 Birkhäuser Verlag, Basel

0013-6018/84/050126-04\$1.50 + 0.20/0

## Berichtigung

**Correction to paper: Packing of 180 equal circles on a sphere.**  
**Elemente der Mathematik. Vol.38, 1983, 119–122**

Professor H.S.M. Coxeter kindly drew my attention to the fact that figures of packings of 72 and 180 circles are chiral and not centro-symmetric. Namely, central symmetry can occur in tessellation  $\{3, q + \}_{b,c}$  ( $q = 4$  or  $5$ ) if  $bc(b - c) = 0$ , that is, the tessellation has a plane of symmetry. Thus, the statement in the last sentence of the paper is not valid.

Tibor Tarnai  
Hungarian Institute for Building Science Budapest