

A further remark concerning Nagel cevians

Autor(en): **Janous, W.**

Objekttyp: **Article**

Zeitschrift: **Elemente der Mathematik**

Band (Jahr): **43 (1988)**

Heft 4

PDF erstellt am: **24.05.2024**

Persistenter Link: <https://doi.org/10.5169/seals-40808>

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Suggestions for further research. We observe that in all cases presented above, as in the real case, the periodic continued fractions are symmetric. Namely, for period length k we have $a_j = a_{k-j}$ and $b_j = b_{k+1-j}$ for $j = 1, 2, \dots, k-1$. Further, $a_k = 2 \cdot a_0$ appears to hold, as in the real case. We found no examples of odd period length.

There is also a connection to Pell-like equations. We illustrate this with the example $\sqrt{376} \in \mathbb{Q}_5$. Let p_n/q_n be the n th convergent of the continued fraction expansion, defined by

$$\begin{aligned} p_{-1} &= 1, \quad p_0 = a_0, \quad p_n = a_n \cdot p_{n-1} + 5^{b_n} \cdot p_{n-2} \quad \text{for } n \geq 1, \\ q_{-1} &= 0, \quad q_0 = 1, \quad q_n = a_n \cdot q_{n-1} + 5^{b_n} \cdot q_{n-2} \quad \text{for } n \geq 1. \end{aligned}$$

Put

$$p_n^2 - 376 \cdot q_n^2 = d_n \cdot 5^{c_n}, \quad c_n \in \mathbb{N}, \quad d_n \in \mathbb{Z}, \quad 5 \nmid d_n.$$

Then we find that $c_n = \sum_{j=0}^n b_j$, and the sequence $\{d_n\}_{n=-1}^{\infty}$ is given by 1, -3, 17, -4, 17, -3, 1, -3, 17, ..., which is symmetric. The fifth convergent $p_5/q_5 = 12103/603$, for which $d_5 = 1$, gives rise to a sort of 5-adic fundamental unit $12103 + 603 \cdot \sqrt{376}$, in the sense that $p_{i+6j} + q_{i+6j} \cdot \sqrt{376} = (p_i + q_i \cdot \sqrt{376}) \cdot (p_5 + q_5 \cdot \sqrt{376})^j$ for $i = -1, 0, \dots, 4$, and $j = 0, 1, 2, \dots$. It would be interesting to have a more general theory of these matters.

B. M. M. de Weger, Faculty of Applied Mathematics,
University of Twente, Enschede

REFERENCES

- 1 Bundschuh P.: p -adische Kettenbrüche und Irrationalität p -adischer Zahlen. *Elem. Math.* 32, 36–40 (1977).
- 2 Schneider Th.: Über p -adische Kettenbrüche. *Symposia Math.* Vol. IV, 181–189 (1970).
- 3 de Weger B. M. M.: Approximation Lattices of p -adic Numbers. *J. Number Th.* 24, 70–88 (1986).

A further remark concerning Nagel cevians

Dedicated to Professor D. S. Mitrinović on his 80th birthday

In El. Math., vol. 41/5 and 42/4, R. H. Eddy and D. S. Milošević considered the cases $L = 14$ and (the improvement) $L = 10$, resp., of

$$\sum n_a \leq 9r + L(R - 2r). \tag{1}$$

Here n_a, n_b, n_c are the Nagel cevians and R, r the circumradius and inradius, resp., of the given triangle.

In this note we show that the minimal L for (1) satisfies

$$6.258998\dots \leq L_{\min} \leq 6.478487\dots$$

The lower bound is $\sup \{F(x), x > 1\}$ with $F(x)$ as given below in i); the upper bound is the largest root of $2x^3 - 3x^2 - 144x + 515 = 0$.

Proof. Let N_a be the foot of n_a on side BC . Then, $BN_a = s - c$, and from the triangle ABN_a it follows via the law of cosines that

$$n_a^2 = c^2 + (s - c)^2 - 2c(s - c)\cos B.$$

After some algebraic manipulations this simplifies to

$$n_a = s \sqrt{(b - c)^2 + 4r^2}/a$$

where s represents the semiperimeter of the given triangle.

i) Considering triangles with $c = 2$ and $a = b = x, x > 1$, we get $r = \sqrt{(x - 1)/(x + 1)}$, $R = x^2/2\sqrt{x^2 - 1}$, $n_c = \sqrt{x^2 - 1}$ and $n_a = n_b = \sqrt{x + 1}\sqrt{(x - 2)^2(x + 1) + 4(x - 1)/x}$. Inserting this in (1) we get after some algebraic manipulations

$$\begin{aligned} L \geq F(x) := & 2\sqrt{x - 1}\{(2x + 2)\sqrt{(x - 2)^2(x + 1) + 4(x - 1)} \\ & + (x^2 - 8x)\sqrt{x - 1}\}/x(x - 2)^2. \end{aligned}$$

Numerical calculations lead to $L \geq F(4.699\dots) = 6.258998\dots =: m$.

ii) The inequality between the arithmetic and square root means yields

$$\begin{aligned} \text{i.e. } \sum n_a &\leq \sqrt{3 \sum n_a^2}, \\ \sum n_a &\leq s \{3 \sum (b^2 + c^2 - 2bc + 4r^2)/a^2\}^{1/2}. \end{aligned} \tag{2}$$

From $\sum a^2 = 2s^2 - 8Rr - 2r^2$ and $\sum bc = r^2 + s^2 + 4Rr$ we get $b^2 + c^2 - 2bc + 4r^2 = 2ab + 2ac - a^2 - 16Rr$. This and (2) yield

$$\sum n_a \leq s \{6 \sum (b + c)/a - 9 - 48Rr \sum 1/a^2\}^{1/2}. \tag{3}$$

We leave it as an exercise to the reader to derive the identities $\sum(b + c)/a = (s^2 + r^2)/2Rr - 1$ and $\sum 1/a^2 = \{(s^2 + 4Rr + r^2)/4Rrs\}^2 - 1/Rr$. Therefore (3) becomes

$$\sum n_a \leq \{(9Rs^2 - 3rs^2 - 48R^2r - 24Rr^2 - 3r^3)/R\}^{1/2}. \tag{4}$$

Let $M = 2L - 9$. For (1) we now consider

$$\begin{aligned} \text{i.e. } 9Rs^2 - 3rs^2 - 48R^2r - 24Rr^2 - 3r^3 &\leq L^2R^3 - 2LMR^2r + M^2Rr^2, \\ s^2 &\leq \{L^2R^3 + (48 - 2LM)R^2r + (M^2 + 24)Rr^2 + 3r^3\}/(9R - 3r) =: A. \end{aligned} \tag{5}$$

From [1], item 5.10, the inequality

$$s^2 \leq 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr} =: B$$

is known. Coupling it with (5) we now have to study $B \leq A$, i.e.

$$\begin{aligned} 6(R - 2r)(3R - r)\sqrt{R^2 - 2Rr} &\leq (L^2 - 18)R^3 - (36 + 2LM)R^2r + (M^2 + 63)Rr^2 \\ &= R\{(L^2 - 18)R^2 - (4L^2 - 18L + 36)Rr + (4L^2 - 36L + 144)r^2\}. \end{aligned}$$

Dividing the last inequality by $(R - 2r)\sqrt{R} \geq 0$ we arrive at

$$6(3R - r)\sqrt{R - 2r} \leq \sqrt{R}\{(L^2 - 18)R - (2L^2 - 18L + 72)r\}. \quad (6)$$

Because of $R \geq 2r$ ([1], item 5.1) the right hand side of (6) is non-negative.

Squaring (6) we get upon letting $u = L^2 - 18$ and $v = 2L^2 - 18L + 72$

$$324R^3 - 864R^2r + 468Rr^2 - 72r^3 \leq u^2R^3 - 2uvR^2r + v^2Rr^2,$$

i.e.

$$f(t) := (u^2 - 324)t^3 + (864 - 2uv)t^2 + (v^2 - 468)t + 72 \geq 0,$$

where we put $t = R/r (\geq 2)$.

It is easily checked that $f(2) = 2(v - 2u)^2 \geq 0$. Furthermore, f attains its local minimum at $t_m = (-q + \sqrt{q^2 - 3pr})/3p$, where we set $p = u^2 - 324$, $q = 864 - 2uv$ and $r = v^2 - 468$.

We claim: $q^2 - 3pr \geq 0$ for $L \geq m$.

Indeed,

$$\begin{aligned} q^2 - 3pr &= u^2v^2 + 1404u^2 + 972v^2 - 3456uv + 291\,600 \\ &= (uv - 540)^2 + 108u^2\{9(v/u)^2 - 22(v/u) + 13\} \\ &\geq (uv - 540)^2 - 48u^2 \end{aligned}$$

where we used $g(t) := 9t^2 - 22t + 13 \geq -4/9$ for $t \in \mathbb{R}$. Furthermore, from $v \geq 36$ we obtain $(uv - 540)^2 - 48u^2 \geq 48\{27(u - 15)^2 - u^2\} \geq 0$ if $u(3\sqrt{3} - 1) \geq 45\sqrt{3}$, i.e. $L \geq \{18 + 45\sqrt{3}/(3\sqrt{3} - 1)\}^{1/2} = 6.0477\dots$

As it seems hopeless to get bounds for L from the inequality $f(t_m) \geq 0$ we restrict ourselves to $t_m \leq 2$ (and we are done because of $f(2) \geq 0$), i.e.

$$\sqrt{q^2 - 3pr} \leq 6p + q. \quad (7)$$

In order to square (7) we have to assure ourselves of $6p + q \geq 0$, i.e.

$$6u^2 - 2uv - 1080 \geq 0, \text{ i.e. } (L^2 - 18)(L^2 + 18L - 126) \geq 540, \text{ i.e. } L \geq 6.25\dots$$

For these values of L (7) becomes

$$\begin{aligned} 12p + 4q + r &\geq 0, \text{ i.e. } 12u^2 - 8uv + v^2 \geq 900, \text{ i.e.} \\ (L-6)(2L^2+9L-90) &\geq 25, \text{ i.e. } L \geq 6,47\ldots \end{aligned}$$

But this was to be shown.

Added in proof. There is now high numerical evidence for the conjecture:

$$L_{\min} = 6.258998\ldots$$

Remarks. 1) It should be noted that in El. Math., vol. 35/5, R. H. Eddy proved the inequality

$$\sum n_a \geq \sum m_a$$

where m_a , m_b , m_c denote the medians of the given triangle. This and [1], item 8.3, i.e.

$$\sum m_a \geq 9r$$

yield the following converse of (1)

$$\sum n_a \geq 9r.$$

2) Inequality (4) and [1], item 5.9, i.e.

$$s^2 \leq 4R^2 + 4Rr + 3r^2$$

immediately lead to

$$\sum n_a < 6R - 2r.$$

3) It remains an open question to determine the precise value of L_{\min} .

W. Janous, Ursulinengymnasium, Innsbruck

REFERENCE

- [1] Bottema O. et al.: Geometric Inequalities. Groningen 1968.