A further remark concerning Nagel cevians

Suggestions for further research. We observe that in all cases presented above, as in the real case, the periodic continued fractions are symmetric. Namely, for period length k we have $a_j = a_{k-j}$ and $b_j = b_{k+1-j}$ for $j = 1, 2, \ldots, k-1$. Further, $a_k = 2 \cdot a_0$ appears to hold, as in the real case. We found no examples of odd period length.
There is also a connection to Pell-like equations. We illustrate this with the example $\sqrt{376} \in \mathbb{Q}_5$. Let p_n/q_n be the nth convergent of the continued fraction expansion, defined by

$$p_{-1} = 1,\ p_0 = a_0,\ p_n = a_n \cdot p_{n-1} + 5^{b_n} \cdot p_{n-2} \quad \text{for } n \geqq 1,$$
$$q_{-1} = 0,\ q_0 = 1,\ q_n = a_n \cdot q_{n-1} + 5^{b_n} \cdot q_{n-2} \quad \text{for } n \geqq 1.$$

Put

$$p_n^2 - 376 \cdot q_n^2 = d_n \cdot 5^{c_n},\ c_n \in \mathbb{N},\ d_n \in \mathbb{Z},\ 5 \nmid d_n.$$

Then we find that $c_n = \sum_{j=0}^{n} b_j$, and the sequence $\{d_n\}_{n=-1}^{\infty}$ is given by $1, -3, 17, -4, 17, -3, 1, -3, 17, \ldots$, which is symmetric. The fifth convergent $p_5/q_5 = 12\,103/603$, for which $d_5 = 1$, gives rise to a sort of 5-adic ‹fundamental unit› $12\,103 + 603 \cdot \sqrt{376}$, in the sense that $p_{i+6j} + q_{i+6j} \cdot \sqrt{376} = (p_i + q_i \cdot \sqrt{376}) \cdot (p_5 + q_5 \cdot \sqrt{376})^j$ for $i = -1, 0, \ldots, 4$, and $j = 0, 1, 2, \ldots$. It would be interesting to have a more general theory of these matters.

B. M. M. de Weger, Faculty of Applied Mathematics,
University of Twente, Enschede

REFERENCES

1 Bundschuh P.: p-adische Kettenbrüche und Irrationalität p-adischer Zahlen. Elem. Math. *32*, 36–40 (1977).
2 Schneider Th.: Über p-adische Kettenbrüche. Symposia Math. Vol. *IV*, 181–189 (1970).
3 de Weger B. M. M.: Approximation Lattices of p-adic Numbers. J. Number Th. *24*, 70–88 (1986).

 0013-6018/88/040112-05$1.50+0.20/0

A further remark concerning Nagel cevians

Dedicated to Professor D. S. Mitrinović on his 80th birthday

In El. Math., vol. 41/5 and 42/4, R. H. Eddy and D. S. Milošević considered the cases $L = 14$ and (the improvement) $L = 10$, resp., of

$$\sum n_a \leq 9r + L(R - 2r). \tag{1}$$

Here n_a, n_b, n_c are the Nagel cevians and R, r the circumradius and inradius, resp., of the given triangle.

In this note we show that the minimal L for (1) satisfies

$$6.258998\ldots \leq L_{\min} \leq 6.478487\ldots.$$

The lower bound is $\sup\{F(x), x > 1\}$ with $F(x)$ as given below in i); the upper bound is the largest root of $2x^3 - 3x^2 - 144x + 515 = 0$.

Proof. Let N_a be the foot of n_a on side BC. Then, $BN_a = s - c$, and from the triangle ABN_a it follows via the law of cosines that

$$n_a^2 = c^2 + (s-c)^2 - 2c(s-c)\cos B.$$

After some algebraic manipulations this simplifies to

$$n_a = s\sqrt{(b-c)^2 + 4r^2/a}$$

where s represents the semiperimeter of the given triangle.
i) Considering triangles with $c = 2$ and $a = b = x$, $x > 1$, we get $r = \sqrt{(x-1)/(x+1)}$, $R = x^2/2\sqrt{x^2-1}$, $n_c = \sqrt{x^2-1}$ and $n_a = n_b = \sqrt{x+1}\sqrt{(x-2)^2(x+1) + 4(x-1)/x}$. Inserting this in (1) we get after some algebraic manipulations

$$\begin{aligned} L \geq F(x) := 2\sqrt{x-1}\,\{&(2x+2)\sqrt{(x-2)^2(x+1) + 4(x-1)} \\ &+ (x^2 - 8x)\sqrt{x-1}\}/x(x-2)^2. \end{aligned}$$

Numerical calculations lead to $L \geq F(4.699\ldots) = 6.258998\ldots =: m$.
ii) The inequality between the arithmetic and square root means yields

$$\sum n_a \leq \sqrt{3\sum n_a^2},$$
i.e.
$$\sum n_a \leq s\{3\sum(b^2 + c^2 - 2bc + 4r^2)/a^2\}^{1/2}. \tag{2}$$

From $\sum a^2 = 2s^2 - 8Rr - 2r^2$ and $\sum bc = r^2 + s^2 + 4Rr$ we get $b^2 + c^2 - 2bc + 4r^2 = 2ab + 2ac - a^2 - 16Rr$. This and (2) yield

$$\sum n_a \leq s\{6\sum(b+c)/a - 9 - 48Rr\sum 1/a^2\}^{1/2}. \tag{3}$$

We leave it as an exercise to the reader to derive the identities $\sum(b+c)/a = (s^2 + r^2)/2Rr - 1$ and $\sum 1/a^2 = \{(s^2 + 4Rr + r^2)/4Rrs\}^2 - 1/Rr$. Therefore (3) becomes

$$\sum n_a \leq \{(9Rs^2 - 3rs^2 - 48R^2r - 24Rr^2 - 3r^3)/R\}^{1/2}. \tag{4}$$

Let $M = 2L - 9$. For (1) we now consider

$$9Rs^2 - 3rs^2 - 48R^2r - 24Rr^2 - 3r^3 \leq L^2R^3 - 2LMR^2r + M^2Rr^2,$$
i.e.
$$s^2 \leq \{L^2R^3 + (48 - 2LM)R^2r + (M^2 + 24)Rr^2 + 3r^3\}/(9R - 3r) =: A. \tag{5}$$

From [1], item 5.10, the inequality

$$s^2 \le 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2 - 2Rr} =: B$$

is known. Coupling it with (5) we now have to study $B \le A$, i.e.

$$6(R-2r)(3R-r)\sqrt{R^2-2Rr} \le (L^2-18)R^3 - (36+2LM)R^2 r + (M^2+63)Rr^2$$

$$= R\{(L^2-18)R^2 - (4L^2-18L+36)Rr + (4L^2-36L+144)r^2\}.$$

Dividing the last inequality by $(R-2r)\sqrt{R} \ge 0$ we arrive at

$$6(3R-r)\sqrt{R-2r} \le \sqrt{R}\{(L^2-18)R - (2L^2-18L+72)r\}. \tag{6}$$

Because of $R \ge 2r$ ([1], item 5.1) the right hand side of (6) is non-negative.

Squaring (6) we get upon letting $u = L^2 - 18$ and $v = 2L^2 - 18L + 72$

$$324R^3 - 864R^2 r + 468Rr^2 - 72r^3 \le u^2R^3 - 2uvR^2 r + v^2Rr^2,$$

i.e.

$$f(t) := (u^2-324)t^3 + (864-2uv)t^2 + (v^2-468)t + 72 \ge 0,$$

where we put $t = R/r$ (≥ 2).
It is easily checked that $f(2) = 2(v-2u)^2 \ge 0$. Furthermore, f attains its local minimum at $t_m = (-q + \sqrt{q^2 - 3pr})/3p$, where we set $p = u^2 - 324$, $q = 864 - 2uv$ and $r = v^2 - 468$.
We claim: $q^2 - 3pr \ge 0$ for $L \ge m$.
Indeed,

$$q^2 - 3pr = u^2v^2 + 1404u^2 + 972v^2 - 3456uv + 291\,600$$

$$= (uv-540)^2 + 108u^2\{9(v/u)^2 - 22(v/u) + 13\}$$

$$\ge (uv-540)^2 - 48u^2$$

where we used $g(t) := 9t^2 - 22t + 13 \ge -4/9$ for $t \in R$. Furthermore, from $v \ge 36$ we obtain $(uv-540)^2 - 48u^2 \ge 48\{27(u-15)^2 - u^2\} \ge 0$ if $u(3\sqrt{3}-1) \ge 45\sqrt{3}$, i.e. $L \ge \{18 + 45\sqrt{3}/(3\sqrt{3}-1)\}^{1/2} = 6.0477\ldots$.
As it seems hopeless to get bounds for L from the inequality $f(t_m) \ge 0$ we restrict ourselves to $t_m \le 2$ (and we are done because of $f(2) \ge 0$), i.e.

$$\sqrt{q^2 - 3pr} \le 6p + q. \tag{7}$$

In order to square (7) we have to assure ourselves of $6p + q \ge 0$, i.e.

$$6u^2 - 2uv - 1080 \ge 0, \text{ i.e. } (L^2-18)(L^2+18L-126) \ge 540, \text{ i.e. } L \ge 6.25\ldots.$$

For these values of L (7) becomes

$$12p + 4q + r \geq 0, \text{ i.e. } 12u^2 - 8uv + v^2 \geq 900, \quad \text{i.e.}$$
$$(L-6)(2L^2 + 9L - 90) \geq 25, \quad \text{i.e.} \quad L \geq 6{,}47\ldots.$$

But this was to be shown.

Added in proof. There is now high numerical evidence for the conjecture:

$$L_{\min} = 6.258998\ldots.$$

Remarks. 1) It should be noted that in El. Math., vol. 35/5, R. H. Eddy proved the inequality

$$\sum n_a \geq \sum m_a$$

where m_a, m_b, m_c denote the medians of the given triangle. This and [1], item 8.3, i.e.

$$\sum m_a \geq 9r$$

yield the following converse of (1)

$$\sum n_a \geq 9r.$$

2) Inequality (4) and [1], item 5.9, i.e.

$$s^2 \leq 4R^2 + 4Rr + 3r^2$$

immediately lead to

$$\sum n_a < 6R - 2r.$$

3) It remains an open question to determine the precise value of $L_{\min}$.

W. Janous, Ursulinengymnasium, Innsbruck

REFERENCE

[1] Bottema O. et al.: Geometric Inequalities. Groningen 1968.

 0013-6018/88/040116-04$1.50+0.20/0