When is a polygon circumscribing a regular polygon again regular?

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When is a Polygon Circumscribing a Regular Polygon Again Regular?

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Introduction

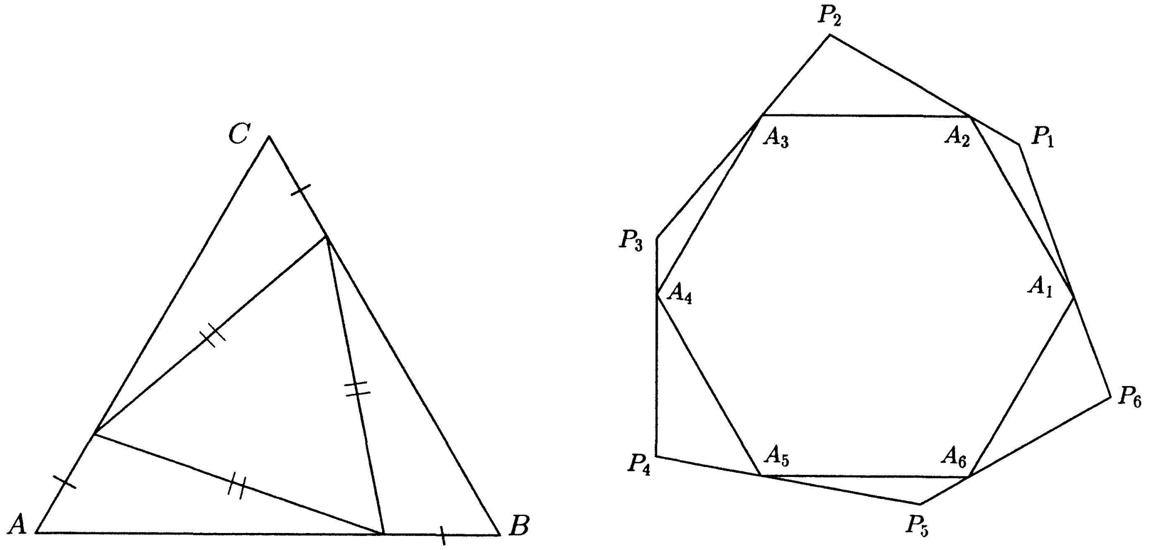
In the March 1970 issue of "Mathematics Magazine" the following problem appeared: Is the triangle $\triangle ABC$ in Fig. 1a equilateral? Four different solutions to this problem appeared in its November 1970 issue, and shortly thereafter the English geometer J.F. Rigby generalized the problem to other polygons [2], [3].[2]) In particular, he showed that the

Gegeben sei ein n-Eck **P** mit Eckpunkten $P_1, P_2, \ldots, P_n$. Auf den Seiten $P_{i-1}P_i$, $i = 1, 2, \ldots, n$ ($P_0 = P_n$) seien Punkte A_i festgelegt mit $A_1P_1 = A_2P_2 = \ldots = A_nP_n$. Ist **P** ein reguläres Polygon, so ist offensichtlich das aus den Punkten $A_1, A_2, \ldots, A_n$ gebildete Polygon **A** ebenfalls regulär. Der vorliegende Beitrag beschäftigt sich mit der Umkehrung dieses Schlusses: Folgt aus der Regularität des Polygons **A** auch die Regularität von **P**? Bekannte Resultate (für $n = 3, 4$, und $n \geq 6$ gerade) werden hier ergänzt und präzisiert. Dabei verdient der Weg, den die Autoren einschlagen, selbständiges Interesse. Das geometrische Problem wird in eine analytische Fragestellung übersetzt, die im Rahmen dynamischer Systeme interpretiert wird: Es ist die Frage zu beantworten, ob eine gegebene reelle Funktion einen periodischen Punkt, d.h. ob das zugehörige dynamische System eine periodische Bahn besitzt. In dieser Interpretation lassen sich schliesslich auch experimentell Informationen für den Fall $n = 5$ gewinnen, der in diesem Problem eine merkwürdige Sonderrolle zu spielen scheint. *ust*

1) This work is partially supported by California State University, Long Beach.

2) Rigby also treated other types of "circumscribed polygons" and the analogous question for the hyperbolic plane.

quadrilateral circumscribing the square must itself be a square, but that there exists a non-regular circumscribing hexagon (Fig. 1b).



a) b)

Fig. 1 The circumscribing hexagon has "period 2" (see Th. 1). The angles of $\triangle A_i P_i A_{i+1}$ are 30°, 100°, 50° for i even and 10°, 140°, 30° for i odd.

To be precise, let $\mathbf{A} = A_1 \ldots A_n$ and $\mathbf{P} = P_1 \ldots P_n$ (A_i and P_i labelled counterclockwise) be convex n-sided polygons (abbr. n-gons). Suppose that **P** is circumscribed about **A** (i.e., A_{i+1} lies between P_i and P_{i+1}) with $A_i P_i = A_{i+1} P_{i+1}$, Fig. 1b (we shall refer to **P** as a *circumscribing polygon*). This paper is concerned with the question: if **A** is regular, is **P** necessarily regular?

Besides the cases already mentioned, Rigby proved that the answer is negative for $n \geq 6$ even. In this paper we answer the question for the remaining values of n (Theorem 1, part 1). In addition, we analyze the types of non-regular polygons that arise (Theorem 1, part 2).

The main results of this paper are given in the following theorem.

Theorem 1 *Suppose a regular n-gon* **A** *with sides equal to one is inscribed in an n-gon* **P** *with $A_1 P_1 = A_2 P_2 = \cdots = A_n P_n$. Then*

1. *if $n = 3, 4$ or $n \geq 7$ odd, then* **P** *must be regular;*

2. *if $n \geq 6$ even, then* **P** *may be non-regular. Moreover, for $n \geq 8$, the angles and sides of* **P** *must have period 2, i.e.: $\angle P_1 = \angle P_3 = \cdots$ and $\angle P_2 = \angle P_4 = \cdots$; $P_1 P_2 = P_3 P_4 = P_5 P_6 = \cdots$ and $P_2 P_3 = P_4 P_5 = \cdots$*

Our approach differs markedly from Rigby's: a new theme of *abundant* and *deficient* angles unifies our proof of part 1 of Theorem 1. The basic idea here is quite simple. It is shown in Lemma 2 that for $n \geq 6$ the angles of a non-regular circumscribing n-gon **P** must alternate between being larger than (abundant) and smaller than (deficient) the angle of a regular n-gon, and thus can only occur if n is even. A similar argument works for $n = 3$ and 4.

To prove the second part of Theorem 1 we translate the geometric question (whether there are non-regular n-gons of period k) to an analytic one, namely: Does a certain function f have a periodic point with period equal to k ($k > 1$ a divisor of n)? In Theorem 2 the existence of period 2 points is proved, and in Theorem 3 it is shown that there are no points of period greater than 2 if $n > 6$.

The paper is divided into two sections which contain the proofs of the first and second parts of Theorem 1, respectively. The dynamics of the polygons **P** are discussed in Remark 3, and a bifurcation diagram is given for the case of the hexagon (Fig. 6). A picture of a non-regular pentagon along with graphs are given in Remark 4 at the end of the paper.

1 The Cases in Which There are Only Regular Circumscribing Polygons

Let **A** and **P** be as in Theorem 1. Let α denote $\frac{(n-2)\pi}{n}$, the angle measure of a regular n-gon. Central to our discussion is the following notion for the angles of **P**: the angle $\angle P_i$ is *abundant* (resp. *deficient*) if $\angle P_i > \alpha$ (resp. $\angle P_i < \alpha$).

First we observe that if $\angle P_i = \alpha$, for $1 \leq i \leq n$, then **P** must be regular. (The triangle case is clear. If $n \geq 4$, noting that α is not acute we have $\triangle A_1P_1A_2 \cong \triangle A_2P_2A_3$. Hence $P_1A_2 = P_2A_3$ and $P_1P_2 = P_1A_2 + A_2P_2 = P_2A_3 + A_3P_3 = P_2P_3$.) Thus a *non-regular* circumscribing polygon **P** must have both deficient and abundant angles since the sum of its angles is $n\alpha$.

Geometrically it is easy to determine whether $\angle P_i$ is abundant or deficient. Given two points A and B, the locus of points P with $\angle APB$ equal to a given angle measure are two circular arcs, one on each side of the segment $\overline{AB}$. When A, B are A_i, A_{i+1}, and the given angle measure is α, the arc which lies *outside* (= the side opposite the center of **A**) of the segment $\overline{A_iA_{i+1}}$ will be referred to as the *regular arc* on $\overline{A_iA_{i+1}}$ and denoted by $\widehat{A_iA_{i+1}}$. (See Figs. 2, 3, and 4.) If P_i lies *inside* (resp. *outside*) the regular arc $\widehat{A_iA_{i+1}}$ then $\angle P_i > \alpha$ (resp. $\angle P_i < \alpha$). Thus $\angle P_i$ is *abundant* (resp. *deficient*) if and only if P_i lies *inside* (resp. *outside*) the regular arc $\widehat{A_iA_{i+1}}$.

Next we show that a regular arc and its neighboring sides are tangent. Suppose $n \geq 5$, then the rays $\overrightarrow{A_{i-1}A_i}$ and $\overrightarrow{A_{i+2}A_{i+1}}$ intersect, say at Q. Suppose $\overrightarrow{OQ}$ intersects the regular arc $\widehat{A_iA_{i+1}}$ at R_i, where O is the center of the circle containing the regular arc $\widehat{A_iA_{i+1}}$ (Fig. 2).

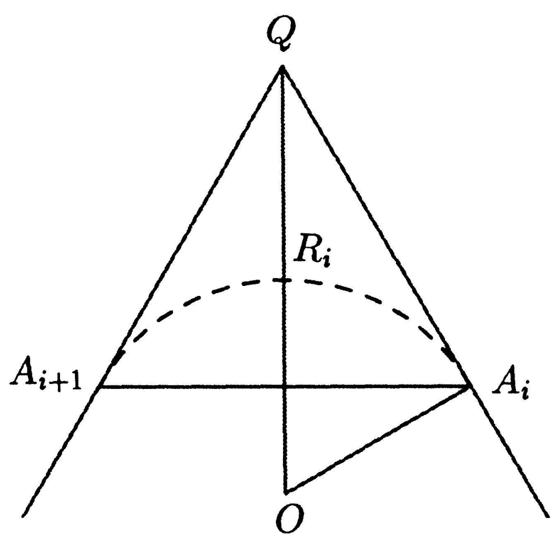


Fig. 2 ($n = 6$)

Then

$$\angle A_iQA_{i+1} = \pi - 2\angle QA_iA_{i+1} = \pi - 2(\pi - \alpha) = 2\alpha - \pi = \frac{n-4}{n}\pi. \qquad (1)$$
$$\angle A_iOR_i = \pi - 2\angle A_iR_iO = \pi - \angle A_iR_iA_{i+1} = \pi - \alpha.$$

Using these two equations we get

$$\angle OA_iQ = \pi - \frac{1}{2}\angle A_iQA_{i+1} - \angle A_iOR_i = \pi - (\alpha - \frac{\pi}{2}) - (\pi - \alpha) = \frac{\pi}{2}.$$

This proves the following lemma for $n \geq 5$:

Lemma 1 *Each regular arc $\widehat{A_iA_{i+1}}$ is tangent to its two neighboring sides $\overline{A_{i-1}A_i}$ and $\overline{A_{i+1}A_{i+2}}$ at A_i and A_{i+1}, respectively.*

In fact it is easy to check that this Lemma also holds for $n = 3$ and 4.

We make a simple but crucial observation (needed in the proof of the following key Lemma 2). Suppose R_i lies on the regular arc $\widehat{A_iA_{i+1}}$. Join R_i to A_{i+1} and extend it until it meets the next regular arc $\widehat{A_{i+1}A_{i+2}}$, say at R_{i+1} (Fig. 3a). Then $A_iR_i = A_{i+1}R_{i+1}$. (Proof. Since $\angle A_{i+1} = \alpha = \angle R_i$ implies $\angle A_{i+1}A_iR_i = \angle A_{i+2}A_{i+1}R_{i+1}$, we get by the side-angle-angle theorem that $\triangle A_iR_iA_{i+1} \cong \triangle A_{i+1}R_{i+1}A_{i+2}$.)

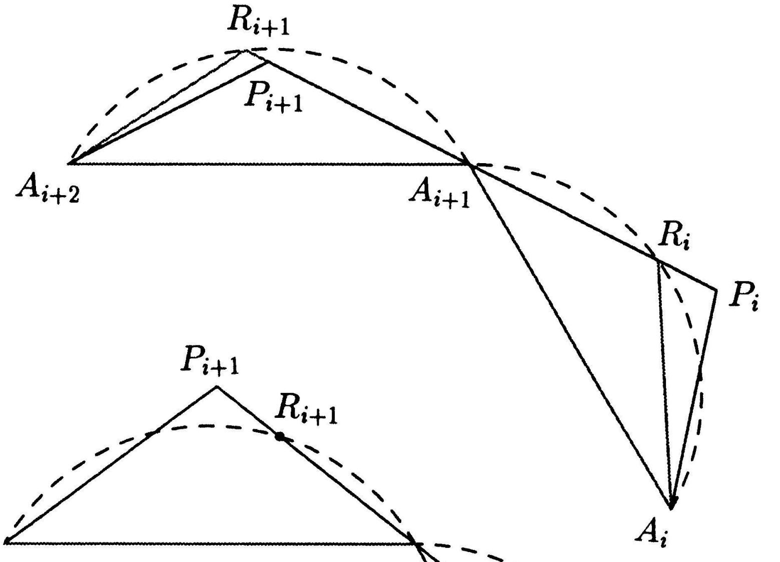


Fig. 3 a)

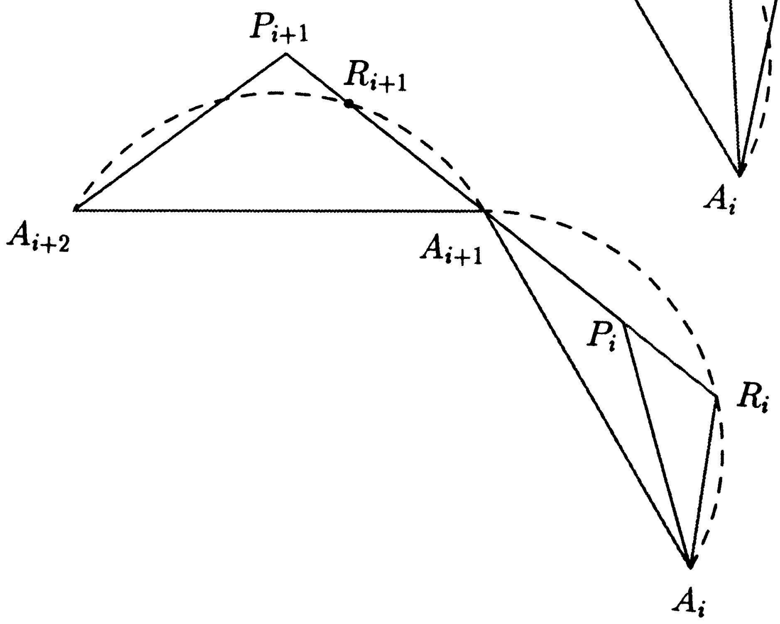


Fig. 3 b)

Lemma 2 *Let* **P** *be a circumscribing n-gon with $n \geq 6$. If $\angle P_i$ is deficient (resp. abundant) then $\angle P_{i+1}$ is abundant (resp. deficient).*

Proof. Since **P** is convex, the vertex P_i lies inside $\triangle A_i A_{i+1} Q$ (Fig. 2). In view of Eq. (1), $\angle P_i > \frac{n-4}{n}\pi$. On the other hand, if $\angle P_i$ is deficient then P_i lies outside the regular arc $\widehat{A_i A_{i+1}}$ (Fig. 3a). Thus the line segment $\overline{P_i A_{i+1}}$ intersects $\widehat{A_i A_{i+1}}$, say at R_i. We have $\angle A_i R_i P_i = \pi - \alpha = \frac{2\pi}{n}$ which is $\leq \frac{(n-4)\pi}{n}$ if $n \geq 6$. Thus $\angle P_i > \angle A_i R_i P_i$ and $A_i P_i < A_i R_i$. Construct R_{i+1} from R_i as in the above observation. Since $A_i R_i = A_{i+1} R_{i+1}$, the last inequality becomes $A_{i+1} P_{i+1} < A_{i+1} R_{i+1}$. Since A_{i+1}, R_{i+1} and P_{i+1} are collinear (they all lie on the line $\overleftrightarrow{P_i A_{i+1}}$), P_{i+1} lies between A_{i+1} and R_{i+1}. So P_{i+1} lies inside $\widehat{A_{i+1} A_{i+2}}$ and $\angle P_{i+1}$ is abundant.

The case that $\angle P_i$ is abundant can be argued similarly (see Fig. 3b): We now have P_i is between R_i and A_{i+1} (R_i, R_{i+1} again lie on regular arcs) and $\angle R_i = \alpha > \frac{\pi}{2} > \angle A_i P_i R_i$, which implies $A_i R_i < A_i P_i$ or $A_{i+1} R_{i+1} < A_{i+1} P_{i+1}$. Thus R_{i+1} is between A_{i+1} and P_{i+1} and $\angle P_{i+1}$ is deficient.

Remark 1. A similar argument yields: If $\angle P_i$ is *deficient* then $\angle P_{i+1}$ is deficient if $n = 3$ or 4, and can be either abundant or deficient if $n = 5$. On the other hand, if $\angle P_i$ is *abundant* then $\angle P_{i+1}$ is abundant if $n = 3$ and deficient if $n = 4$ or 5.

Proof of Theorem 1, part 1: if $n = 3, 4$ or $n \geq 7$ odd, then any circumscribing n-gon **P** *must be regular.*

Suppose **P** is a *non-regular* circumscribing polygon. By the observation at the beginning of this section, at least one $\angle P_i$ is deficient. If furthermore $n = 3$ or 4, by remark 1, the existence of one deficient angle would imply that all angles are deficient, a contradiction. If on the other hand $n \geq 6$ then, by Lemma 2, the existence of one deficient angle implies that the angles alternate between abundant and deficient, which can happen only if n is even.

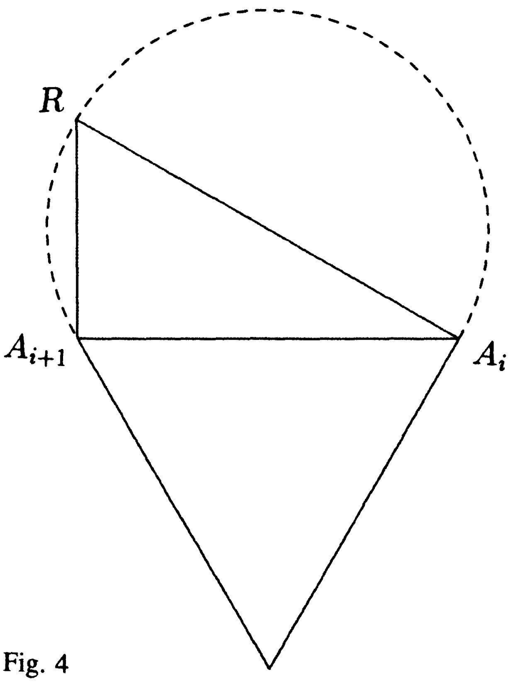


Fig. 4

We conclude this section with a remark about the range of the length $\ell := A_i P_i$.

Remark 2. For a circumscribing polygon **P**, regular or not, at least one $\angle P_i$ is $\geq \alpha$, which is $\geq \frac{\pi}{2}$ if $n \geq 4$. By considering $\triangle A_i P_i A_{i+1}$, we see that $0 < \ell = A_i P_i < A_i A_{i+1} = 1$. For $n = 3$ the largest possible value of ℓ is the diameter $A_i R = \frac{2}{\sqrt{3}}$ of the circle containing the regular arc, see Fig. 4.

Note that for each ℓ in the above range a *regular* circumscribing n-gon **P** can be obtained by starting with R_1 on the regular arc with $A_1 R_1 = \ell$ and then construct $R_2, \ldots, R_n$ as in the observation before Lemma 2.

Thus there is a one-to-one correspondence between points on a regular arc and regular polygons **P**.

2 The Cases of Non-Regular Circumscribing Polygons

In this section we will prove part 2 of Theorem 1. That is, it will be shown analytically that whenever $n \geq 6$ is even there are non-regular n-gons $\mathbf{P}$ satisfying the conditions of Theorem 1, and that, with the exception of the hexagon, all these polygons have period 2. In [2] and [3] a geometric proof is outlined for the existence of non-regular polygons, but our analytic approach leads to a description of the types of non-regular polygons $\mathbf{P}$. We adopt the same notation as in section 1. Henceforth, $n \geq 6$, so by Remark 2 the length $\ell := A_iP_i$ takes values in $(0, 1)$.

We consider the function f which for given ℓ computes θ_{i+1} from the argument θ_i, where $\theta_i = \angle A_{i+1}A_iP_i$. A periodic orbit of f of period k corresponds to a polygon $\mathbf{P}$ of period k. More precisely, if ϕ is an angle between 0 and $2\pi/n$ such that the ith iterates $f^i(\phi) > 0$ are distinct for $i = 0, \ldots, k-1$, $f^k(\phi) = \phi$, and n is divisible by k, then the sides and angles of the polygon $\mathbf{P}$ with $\angle A_{i+1}A_iP_i = f^i(\phi)$, $i = 1, \ldots, n$, have period k. (This is a consequence of basic theorems on triangles such as the side-angle-side theorem.)

Let $\psi_i = \angle A_iA_{i+1}P_i$. Then $\theta_{i+1} = \pi - \alpha - \psi_i = 2\pi/n - \psi_i$, where

$$\psi_i = \arctan\left(\frac{P_iT}{A_iA_{i+1} - A_iT}\right) = \arctan\left(\frac{\ell \sin\theta_i}{1 - \ell\cos\theta_i}\right),$$

and T is the foot of the perpendicular of P_i onto the line $\overleftrightarrow{A_iA_{i+1}}$. Therefore, we have

$$f(\theta) = \frac{2\pi}{n} - \arctan\left(\frac{\ell\sin\theta}{1 - \ell\cos\theta}\right).$$

By Remark 2 there is exactly one regular polygon $\mathbf{P}$ for each ℓ. For ℓ fixed, let β represent the angle $\angle A_2A_1P_1$ of this regular polygon. Note that $f(\beta) = \beta$.

Lemma 3 $f'(\beta) < -1$ *if and only if* $\ell > \ell_0$, *and* $f'(\beta) = -1$ *only if* $\ell = \ell_0$, *where*

$$\ell_0 := \left(4 - 3\sin^2\frac{2\pi}{n}\right)^{-\frac{1}{2}}. \tag{2}$$

Proof. Let S be the foot of the perpendicular of A_2 onto the line $\overleftrightarrow{A_1P_1}$ in the case in which $\mathbf{P}$ is regular. Since $n \geq 6$, P_1 lies between A_1 and S. Then

$$\ell = A_1S - P_1S = \cos\beta - c\sin\beta, \tag{3}$$

where $c = \cot(\angle A_2P_1S) = \cot(\pi - \alpha) = \cot(2\pi/n)$.

The derivative of f is given by

$$f'(\theta) = \frac{\ell^2 - \ell\cos\theta}{1 - 2\ell\cos\theta + \ell^2}. \tag{4}$$

Since $1 - 2\ell\cos\theta + \ell^2 > 0$, we have $2\ell^2 - 3\ell\cos\beta + 1 < 0$ whenever $f'(\beta) < -1$. Substituting Eq. (3) into the above inequality leads to $\cot\beta > 2c + 1/c$. Since the cotangent function is decreasing in the first quadrant we get $\beta < \operatorname{arccot}(2c + 1/c) =: \gamma$. But $\cos\theta - c\sin\theta$ is decreasing in the first quadrant, so $\ell > \cos\gamma - c\sin\gamma = (c^2+1)/\sqrt{4c^4 + 5c^2 + 1} = \sqrt{1+c^2}/\sqrt{1+4c^2}$ which simplifies to ℓ_0. Since the argument is valid if all the inequality signs are reversed (or replaced by equal signs) the assertion is proved.

Lemma 4 *Let $h(\theta) = f^2(\theta) - \theta$, where f^2 is the composition of f with itself. Then $h'(\beta) > 0$ if $\ell > \ell_0$, and $h'(\beta) < 0$ if $\ell < \ell_0$.*

Proof. Since $h'(\theta) = f'(f(\theta))f'(\theta) - 1$, we get $h'(\beta) = [f'(\beta)]^2 - 1$. Therefore, by the previous lemma, it suffices to show that $f'(\beta) < 1$. But this is equivalent to $\ell \cos\beta < 1$.

We are now ready to establish the second part of Theorem 1: the next theorem deals with period 2 circumscribing polygons, whereas Theorem 3 deals with higher periods.

Theorem 2 *Let $n \geq 6$ be even. Then there exists a non-regular circumscribing polygon of period 2 for each ℓ, $\ell_0 < \ell < \ell_1$, where $\ell_1 := (2\cos(2\pi/n))^{-1}$, and ℓ_0 is defined in Eq. (2).**)

Based on plots of f^2, we conjecture that for each $\ell \in (\ell_0, \ell_1)$, there is *only* one period 2 circumscribing polygon, and for $\ell \notin (\ell_0, \ell_1)$, there are no period 2 circumscribing polygons (see Fig. 5). The complexity of f^2 prevents us from obtaining an analytic proof of this conjecture.

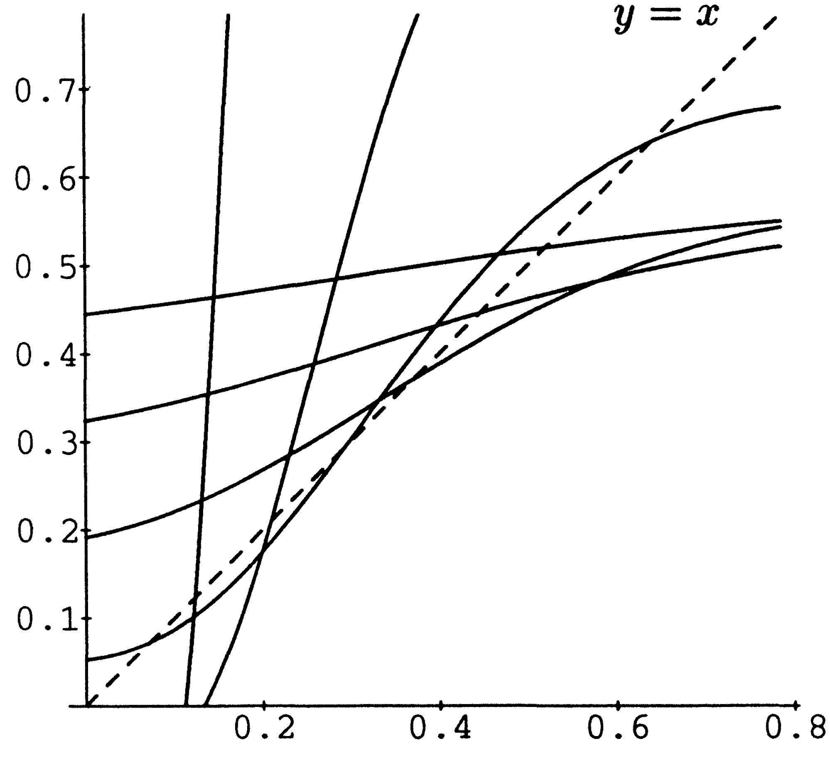


Fig. 5 ($n = 8$, $\ell_0 \approx 0.632$, $\ell_1 \approx 0.707$) Graphs of f^2 for $\ell = .37, .47, .57, .67, .77, .87$. The steeper graphs correspond to larger values of ℓ.

Proof of Theorem 2. Since $h(\theta) = f^2(\theta) - \theta$, a zero $\phi > 0$ of h other than β corresponds to a non-regular polygon **P** of period 2 as long as $f(\phi) > 0$. By the definition of β,

*) ℓ_1 can be interpreted geometrically as A_iQ, where Q is defined in Fig. 2. Also a simple check verifies that $\ell_0 < \ell_1$. Moreover, both $\{\ell_0(n)\}$ and $\{\ell_1(n)\}$ are decreasing sequences which converge to one-half, and $\ell_1 - \ell_0 = O(n^{-2})$.

$h(\beta) = 0$. Moreover, by Lemma 4, $h'(\beta) > 0$. Therefore, if $h(0) > 0$ then h will have at least one zero in the interval $(0, \beta)$ (by the intermediate value theorem). However, $h(0) = f(p)$, $p = 2\pi/n$, so $h(0) > 0$ is equivalent to $\tan p > \ell \sin p/(1-\ell\cos p)$. But this inequality holds if (and only if) $\ell < \ell_1$. (Note also that $f(\phi) > 0$ since $\angle P_1$ is abundant, where $\phi = \angle A_2A_1P_1$.)

Remark 3. By examining plots of the function f^2, we are led to conjecture that as ℓ passes through ℓ_0, the iteration goes through a period-doubling bifurcation (see [1], pp.158–159 for a discussion of period-doubling bifurcation). This is illustrated in Fig. 6.

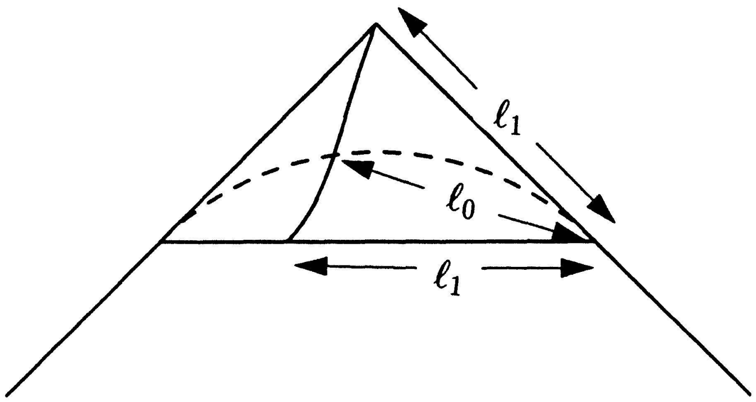


Fig. 6 ($n = 8$) The dotted regular arc represents the vertices of regular polygons for $0 < \ell < 1$; the other curve consists of the two positions (for various ℓ) of vertices of period-two polygons, one on each side of the regular arc. The bifurcation occurs at the intersection of the two curves.

Fig. 7a gives a nonregular hexagon (corresponding to an ℓ near ℓ_0) and the regular hexagon (corresponding to ℓ_0) from which the nonregular hexagon bifurcates. On the other hand, as ℓ increases to $\ell_1 = 1$, the non-regular hexagon approaches an equilateral triangle. This is illustrated in Fig. 7b. In general, the non-regular $2m$-gon approaches a regular m-gon as ℓ increases to ℓ_1.

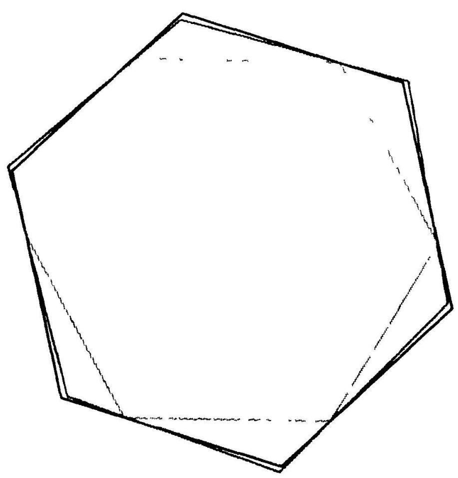


Fig. 7a)

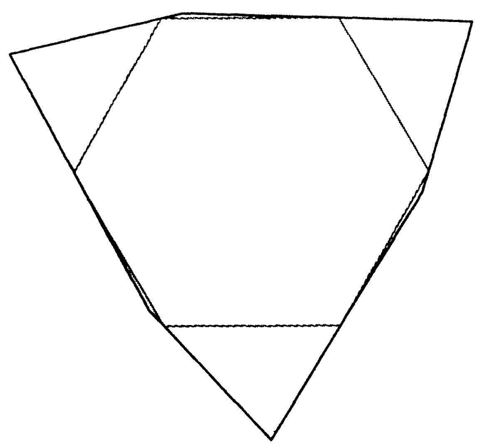


Fig. 7b)

Theorem 3 *There are no non-regular n-gons, $n > 6$, with period greater than two.*

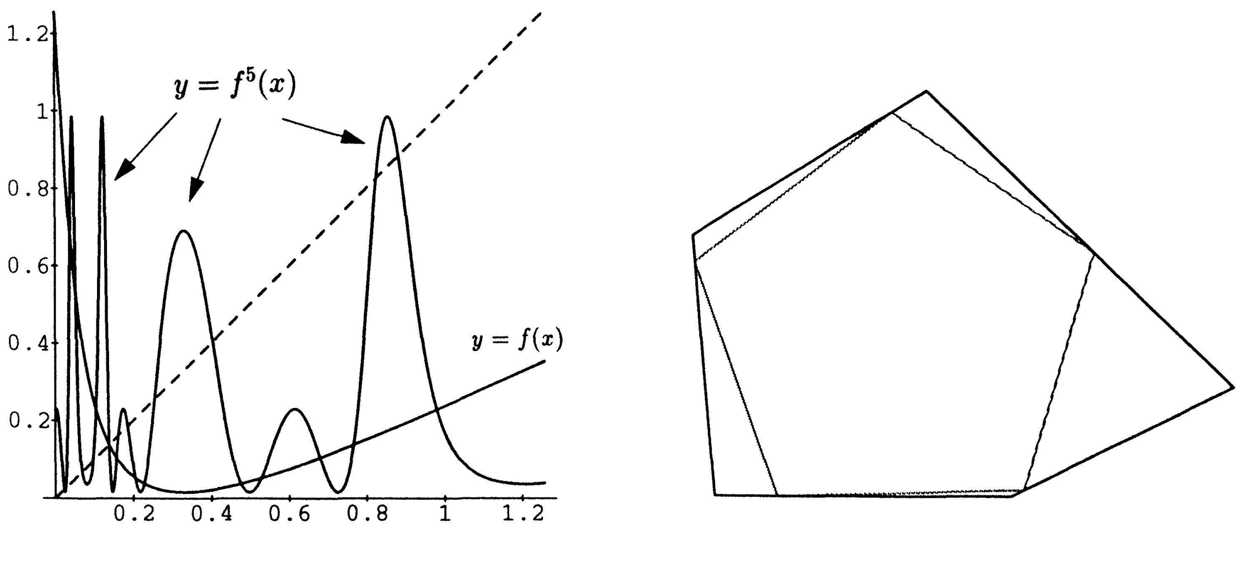
Proof. This is equivalent to there being no orbits of f of period greater than 2 which are contained in the interval $(0, p)$, $p = 2\pi/n$, $n > 6$. Since a decreasing function cannot have a point with period greater than 2, it suffices to show that for $0 < \ell < 1$ f is decreasing on the subset of $(0, p)$ where $f > 0$. In view of Eq. (4) $f'(\theta) \leq 0$ if $\ell \leq \cos\theta$. By the first part of Theorem 1 we have $n \geq 8$, so $p \leq \pi/4$. If $\ell \leq 1/\sqrt{2}$ then $f'(\theta) \leq 0$ on $(0, p)$.

Consider the case $1/\sqrt{2} < \ell < 1$. By the proof of Theorem 2, $f(p) < 0$ if $\ell_1 < \ell < 1$. But $\ell_1 = (2\cos p)^{-1} \leq (2\cos\pi/4)^{-1} = 1/\sqrt{2}$ for $n \geq 8$, so $f(p) < 0$ in this case. However, f is concave up on $(0, p)$ since

$$f''(\theta) = \frac{\sin\theta(\ell - \ell^3)}{(1 - 2\ell\cos\theta + \ell^2)^2}$$

is positive for $0 < \theta < \pi$. Thus, in this case, the concavity of f and the fact that $f(p) < 0$ imply that f is decreasing on the subinterval of $(0, p)$ on which $f > 0$.

Remark 4. The results so far still leave unanswered the question whether there are non-regular pentagons and non-regular hexagons of period 6. (Non-regular hexagons of period 3 are precluded since, by Lemma 2, deficient and abundant angles must alternate.) By examining plots of the function f and its iterates we have concluded that there are non-regular pentagons (Fig. 8) but no non-regular hexagons of period 6 (Fig. 9). Indeed, there seems to be a range of ℓ for each value of which there are *two* non-regular pentagons.



a) b)

Fig. 8 ($n = 5$) Graphs of f and f^5 for $\ell = .946$. Note that f^5 has eleven fixed points so there are two non-regular pentagons for this value of ℓ.

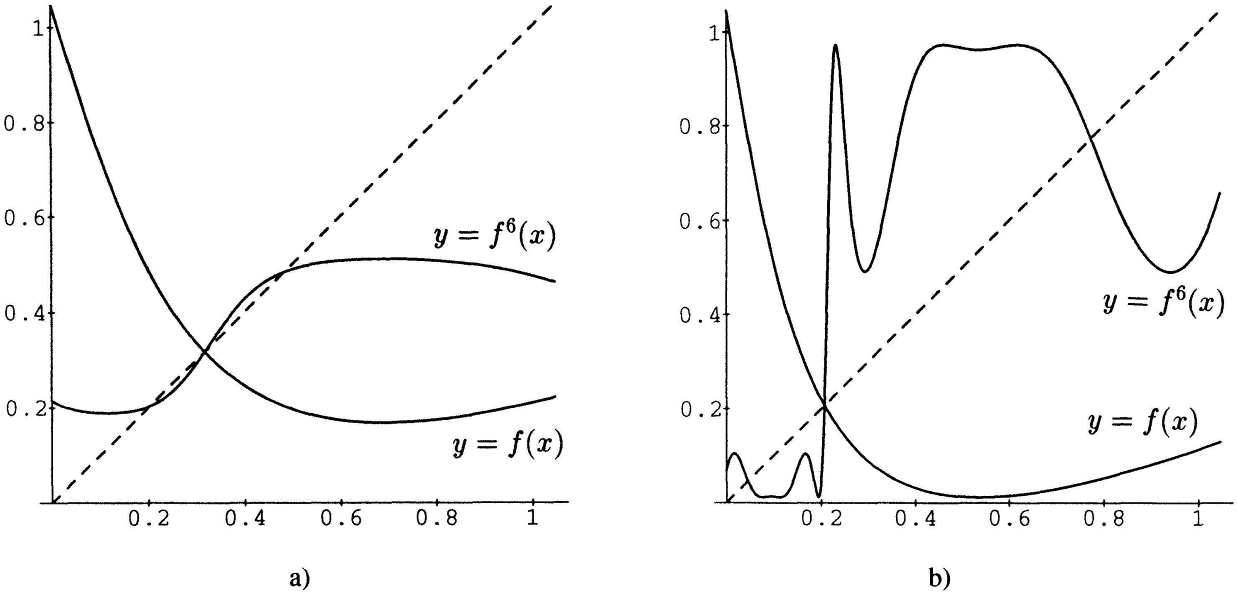


Fig. 9 (n = 6) Graphs of f and f^6 for (a) $\ell = .77$ and (b) $\ell = .86$.

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