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Densest Packing of Six Equal Circles in a Square

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The problem of finding densest packings of n congruent circles inside a compact convex plane region has been investigated thoroughly during the past decades. In particular, a lot of work has been carried out on the determination of optimal circle packings in a circle, a square and an equilateral triangle for small values of n (cf. [2, 3, 16]).

In 1967, circle packings inside a circular disc were given for $n = 2, \dots, 16$ by S. Kravitz ([9]). The optimality of these configurations for $n \leq 7$ was proved by R.L. Graham ([1]), and U. Pirl ([19]) gave proofs for $n \leq 10$. Pirl also made some conjectures for $11 \leq n \leq 19$. Subsequent improvements were obtained for $n = 14, 16, 17$ and 20 ([4]) and again for $n = 17$ ([20]). The optimality for $n = 11$ was proved recently by the author ([11]).

Densest packings of n circles in an equilateral triangle are known for the triangular numbers $n = k(k+1)/2$ (see [17]) and for $n \leq 12$ ([10, 13]). Further conjectures are given in [11, 12].

The problem of optimally packing circles into a square was raised for $n = 8$ by L. Moser ([15]). The optimality of the conjectured packing was proved by J. Schaer and A. Meir ([23]). Schaer ([22]) also solved the problem for $n = 9$ and gave configurations for $n \leq 7$. He remarked that the cases $n = 2, 3, 4$ and 5 'are solved easily', and that $n = 6$

Hans Melissen nimmt hier das Problem des vorhergehenden Beitrages noch einmal auf: Es sollen 6 gleiche Kreise mit möglichst grossem Radius in einem Quadrat plaziert werden. An mehreren Stellen in der Literatur wird die optimale Anordnung für 6 Kreise ohne Beweis und ohne Hinweis auf eine Quelle beschrieben.
Nach der Drucklegung dieses Beitrages hat Hans Melissen allerdings festgestellt, dass B.L. Schwartz 1970 einen entsprechenden Beweis geliefert hat; siehe dazu die Notiz am Ende des Beitrages. *ust*

had been proved by R.L. Graham. A geometric outline of a proof for $n = 7$ exists only in the form of an unpublished manuscript ([21]). Later, G. Wengerodt ([27, 28, 29]) proved the cases $n = 14, 16, 25$, and $n = 36$ was solved by K. Kirchner and G. Wengerodt ([8]). Recently, computer assisted proofs for $10 \le n \le 20$ have been described by Peikert et al. ([7, 18]). For other values of $n \le 27$ candidates for optimal packings have been given in ([5, 14, 24, 25]).

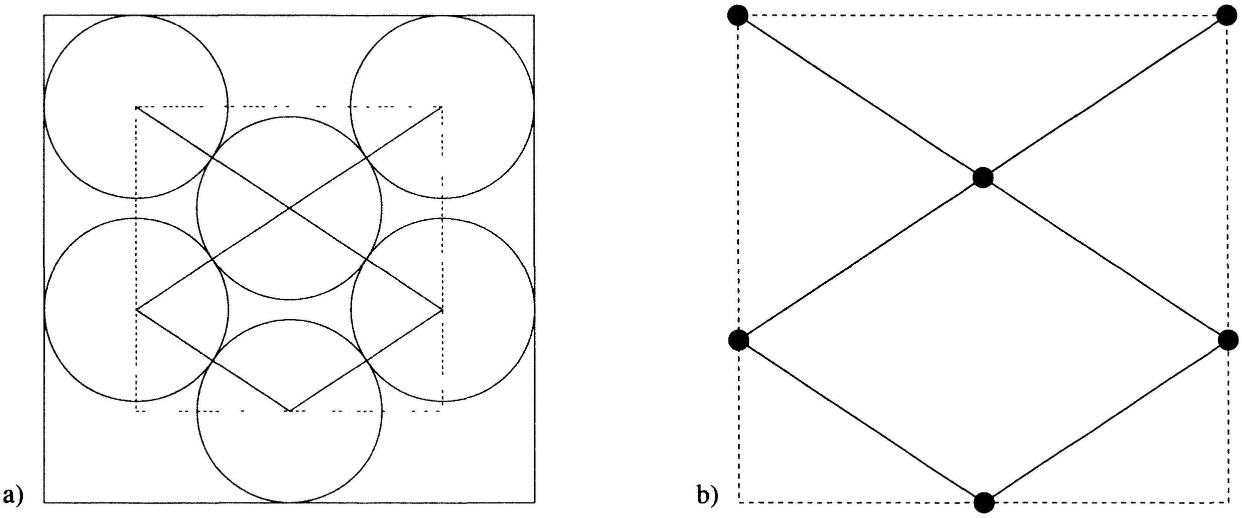

Fig. 1 a) Closest packing of six equal circles in a square.
b) Maximum least distance arrangement for six points in a square. The solid line segments between the points are of equal length.

A useful, often employed fact is that finding a densest packing of n equal circles in a circle, a square or a triangle is equivalent to positioning n points inside that set such that the minimum distance between the points is maximal (see for instance Figure 1a). We shall use this last formulation.

The optimal configuration for six points in a square (up to rotations) is shown in Figure 1b. The minimum distance between the points is $d_6 = \sqrt{13}/6$. The proof of this case was attributed to Graham by Schaer ([22]). It was probably given in a private letter, but, unfortunately, it has never been published, and no further notes exist ([6]). In [2] the desirability of a proof for $n = 6$ was also mentioned. In this paper we will provide such a proof.

The proof is based on the partition of the unit square $[0,1]^2$ into nine smaller regions as indicated in Figure 2. The partition is completely determined by the distances $|p_8 p_{10}| = |p_{10} p_{11}| = 1/3$, $|p_5 p_8| = d_6$ and the obvious symmetries in the diagonals. The diameter of each of the subregions does not exceed d_6. Suppose that we have a configuration $\mathcal{N} = \{x_1, x_2, ..., x_6\}$ of six points in the square for which the minimum distance between the points is equal to $d \ge d_6$. Then each subregion can contain at most one point of this configuration. This is a result of the particular way in which the boundaries are distributed over the subregions, as is indicated by the dashed/solid lines in Figure 2.

First, we note that if there is a point of $\mathcal{N}$ in a B-region as well as points in both of its neighbouring A-regions ($\mathcal{N}$ will then be said to have the '*ABA-property*'), then d is equal to d_6. This can be seen, for instance for A_1, B_1 and A_2, by subdividing the union of these three regions into two regions of diameter d_6 with a cut along $p_1 p_4$ and applying

Dirichlet's pigeon-hole principle (p_1 bisects the lower edge of the square): two of the points must be in the same region, so $d \leq d_6$. It is clear that the points from $\mathcal{N}$ can only be p_1, p_2 and p_3.

We will now consider the two situations in which there is either a point of the configuration in the region C, or $\mathcal{N} \cap C$ is empty.

1. First, suppose that $\mathcal{N} \cap C$ is empty. If there are three or four A-regions containing points from $\mathcal{N}$, then $\mathcal{N}$ has the ABA-property. The remaining alternative is that only two of the A_j each contain a point of the configuration. All four B-regions must then also contain a point. The only situation that is not ABA is where the two A-regions are opposite with respect to C. If, for instance, there is both a point in A_1 and A_3, then the point in A_3 is restricted to a small neighbourhood of p_9, due to the presence of points in B_2 and B_3. This in turn restricts the position of the point in B_2. A similar restriction holds for the point in B_1. It is easy to verify that these two points then lie too close together, so this situation cannot occur.

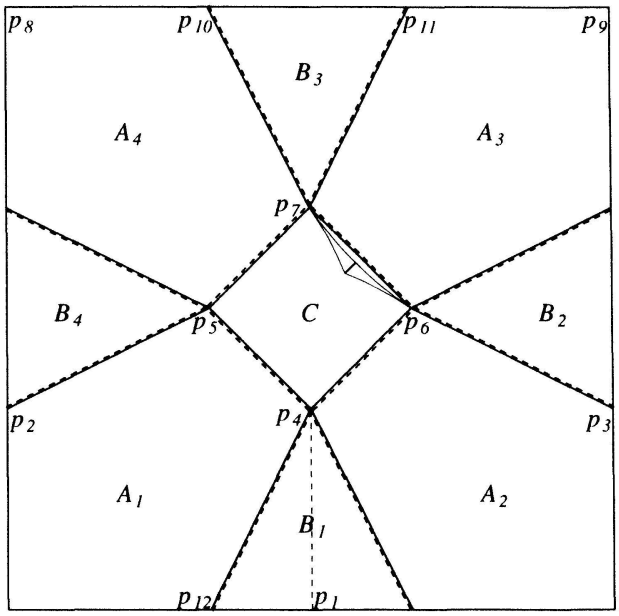

Fig. 2 Partition of the square. The dashed/solid lines indicate to which region each edge is assigned.

2. Secondly, suppose that there is a solution point in region C. It is not possible that two opposite B-regions, like for instance B_1 and B_3, both contain a point of $\mathcal{N}$. This is seen by dividing the union of B_1, C and B_3 with a cut along p_5p_6 into two regions of diameter d_6. It means that at most two B-regions can contain a point of $\mathcal{N}$, so there must be at least three A-regions which contain a point of the configuration. Therefore we either have an ABA-situation, or a situation of the form where there is a point of $\mathcal{N}$ in each of A_2, A_3, A_4, B_1 and B_4. The latter situation is impossible as we will now show.

The three points in the regions A_3, B_1, B_4 restrict the point in C to the small region bounded by three circle segments of radius d around p_2, p_9, p_{12} (see Figure 2). By symmetry, it is sufficient to consider only those positions above the diagonal through p_9. Let (x_j, y_j) $(j = 1, 2, 3, 4)$ denote the coordinates of the points from $\mathcal{N}$ in the regions C, B_4, B_1 and A_2 respectively. The mutual geometric restrictions on the position of these points and the point in A_3 then result in the following inequalities

$$\frac{1}{3} + \sqrt{d^2 - x_1^2} \leq y_1 \leq 1 - \sqrt{d^2 - (1 - x_1)^2}, \tag{1}$$

$$\frac{1}{3} \leq y_2 \leq y_1 - \sqrt{d^2 - x_1^2}, \tag{2}$$

$$\sqrt{d^2 - y_2^2} \le x_3 \le \frac{1}{2}, \tag{3}$$

$$\sqrt{d^2 - (1 - x_3)^2} \le y_4 \le \frac{1}{3}, \tag{4}$$

$$\frac{1}{2} \le x_1 \le 1 - \sqrt{d^2 - (\frac{1 - y_4}{2})^2} \tag{5}$$

The inequalities in (1), for instance, result from the fact that the distances of (x_1, y_1) to p_2 and p_9 should at least be d_9. Initially we have

$$\frac{1}{2} \le x_1 \le 1 - \frac{d}{\sqrt{2}} \le 1 - \frac{d_6}{\sqrt{2}}.$$

Now if we write $x_1 = 1/2 + \varepsilon$, this last inequality leads to $0 \le \varepsilon \le (6 - \sqrt{26})/12$. Some elementary estimates show that inequalities (1) ... (4) imply that

$$y_1 \le \frac{2}{3} - \frac{7}{6}\varepsilon, \quad y_2 \le \frac{1}{3} + \varepsilon, \quad x_3 \ge \frac{1}{2} - \frac{5}{6}\varepsilon, \quad y_4 \ge \frac{1}{3} - \frac{5}{3}\varepsilon.$$

If $d > d_6$, then all the above inequalities are strict. From (5) we see that

$$\frac{1}{2} + \varepsilon = x_1 \le 1 - \sqrt{\frac{1}{4} - \frac{5}{9}\varepsilon - \frac{25}{36}\varepsilon^2} < \frac{1}{2} + \varepsilon,$$

if $\varepsilon > 0$. This shows that the only possible situation occurs for $d = d_6$ and $\varepsilon = 0$. In this case the coordinates of the point in B_4 would be $(0, 1/3)$. By the choice of boundary, however, this point is not in B_4, which shows that this situation is impossible.

We have shown that d_6 is optimal. From the proof it follows that we always end up with an ABA-situation. Suppose for instance that there are three points of $\mathcal{N}$ in A_1, B_1 and A_2. These points can only be p_1, p_2, p_3. There can be no points in B_2 or B_4, so there must be a point in C, because three points in A_3, B_3, A_4 would not be compatible with p_2 and p_3. The only feasible point in C is p_7, so the remaining two points must be p_8, p_9. This results in the solution depicted in Figure 1b. □

Note: After completion of the article it was found that an optimality proof for the packing of six circles in a square has been given previously by Schwartz ([26]). His proof uses similar techniques for a different partition.

References

[1] H.S.M. Coxeter, M.G. Greening and R.L Graham, Sets of points with given maximum separation. Amer. Math. Monthly, 75 (1968), 192–193, (Problem E1921).

[2] H.T. Croft, K.J. Falconer, and R.K. Guy, Unsolved Problems in Geometry, 107–111, Springer Verlag Berlin 1991.

[3] L. Fejes Tóth, Lagerungen in der Ebene, auf der Kugel und im Raum. Springer Verlag Berlin 1972. Second edition.

[4] M. Goldberg, Packing of 14,16,17 and 20 circles in a circle. Math. Mag., 44 (1971), 134–139.

[5] M. Goldberg, The packing of equal circles in a square. Math. Mag., 43 (1970), 24–30.

[6] R.L. Graham, Private communication.

[7] C. de Groot, R. Peikert, and D. Würtz, The optimal packing of ten equal circles in a square. ETH Zürich IPS Research Report No. 90-12 (1990). Submitted.

[8] K. Kirchner and G. Wengerodt, Die dichteste Packung von 36 Kreisen in einem Quadrat. Beiträge Algebra Geom., 25 (1987), 147–159.

[9] S. Kravitz, Packing cylinders into cylindrical containers. Math. Mag., 40 (1967), 65–71.

[10] J.B.M. Melissen, Densest packings of congruent circles in an equilateral triangle. Amer. Math. Monthly., Dec. 1993.

[11] J.B.M. Melissen, Densest packings of eleven congruent circles in a circle. (To appear in Geom. Dedicata.)

[12] J.B.M. Melissen and P.C. Schuur, Packing 16,17 and 18 circles in an equilateral triangle. (To appear in Discrete Math.)

[13] J.B.M. Melissen, Optimal packings of eleven equal circles in an equilateral triangle. Philips Research Manuscript M.S. 17.295 (1992). (Submitted.)

[14] M. Mollard and C. Payan, Some progress in the packing of equal circles in a square. Discrete Math., 84 (1990), 303–307.

[15] L. Moser, Problem 24 (corrected). Can. Math. Bull., 3 (1960), 78.

[16] W.O. Moser and J. Pach, Research Problems in Discrete Geometry. Montreal 1984.

[17] N. Oler, A finite packing problem. Can. Math. Bull., 4 (1961), 153–155.

[18] R. Peikert, D. Würtz, M. Monagan and C. de Groot, Packing circles in a square: A review and new results, Proceedings of the 15th IFIP Conference on System Modelling and Optimization (1991). Springer Lecture Notes in Control and Information Sciences 180, 45–54.

[19] U. Pirl, Der Mindestabstand von n in der Einheitskreisscheibe gelegenen Punkten. Math. Nachr., 40 (1969), 111–124.

[20] G.E. Reis, Dense packings of equal circles within a circle. Math. Mag., 48 (1975), 33–37.

[21] J. Schaer, 1964. Unpublished manuscript, 12 pages. Private communication.

[22] J. Schaer, The densest packing of 9 circles in a square. Can. Math. Bull., 8 (1965), 273–277.

[23] J. Schaer and A. Meir, On a geometric extremum problem. Can. Math. Bull., 8 (1965), 21–27.

[24] K. Schlüter, Kreispackung in Quadraten. Elem. Math., 34 (1979), 12–14.

[25] M. Schmitz and K. Kirchner, Eine Verteilung von 13 Punkten auf einem Quadrat. Wiss. Z. Pädag. Hochschule Erfurt-Mühlhausen, 18 (1982), 113–115.

[26] B.L. Schwartz, Separating points in a square. J. Recr. Math., 3 (1970), 195–204.

[27] G. Wengerodt, Die dichteste Packung von 14 Kreisen in einem Quadrat. Beiträge Algebra Geom., 25 (1987), 25–46.

[28] G. Wengerodt, Die dichteste Packung von 16 Kreisen in einem Quadrat. Beiträge Algebra Geom., 16 (1983), 173–190.

[29] G. Wengerodt, Die dichteste Packung von 25 Kreisen in einem Quadrat. Ann. Univ. Sci. Budapest Eötvös Sect. Math., 30 (1987), 3–15.

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