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Why Means in Two Arguments are Special

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1 Introduction

Starting from the Cauchy-version of the mean value theorem: there exists z between x and y such that

$$\frac{f(x)-f(y)}{g(x)-g(y)}=\frac{f'(z)}{g'(z)}, \tag{1}$$

choosing $f(x) = x^r, g(x) = x^s$, and solving for z, Stolarsky (1975) established the following mean value function:

$$N_{r,s}(x,y)=\left[\frac{s(x^r-y^r)}{r(x^s-y^s)}\right]^{\frac{1}{r-s}} \qquad (r,s\neq 0, r\neq s)\,. \tag{2}$$

$N_{r,s}(x,y)$ has been extensively discussed in Stolarsky (1975, 1980), as well as Leach and Scholander (1978, 1983). The special case of $s = 1$ was introduced in Galvani (1927), already, and Cisbani (1938) investigated the case $s = 0$ (which has to be defined by continuity extension). $N_{r,s}(x,y)$ includes the traditional power mean $M_s(x,y) = \left(\frac{x^s+y^s}{2}\right)^{1/s}$ at $r = 2s$.

We commence by pointing out that generalizations of the power mean M_s to more than two arguments preserve a specific mean value property, whereas generalizations of the more extensive Stolarsky mean $N_{r,s}$ do not.

Die Bildung des harmonischen, geometrischen und arithmetischen Mittelwertes von zwei Zahlen kann auf eine *kontinuierliche* Familie von Mittelwertbildungen verallgemeinert werden. Diese Sachlage hängt offenbar in entscheidender Weise damit zusammen, dass genau zwei Argumente betrachtet werden. Gunter Lorenzen geht in seinem Beitrag zuerst der Frage nach, weshalb hier die Anzahl zwei eine Sonderrolle spielt, und gibt anschliessend einige Anwendungen der verallgemeinerten Mittelwertbildungen. *ust*

2 Generalized Stolarsky means

In order to generalize $N_{r,s}$ to be able to handle more than two arguments, we should start from a generalized mean value theorem. Exploitation of the concept of generalized divided differences (Nörlund (1924), Milne-Thomson (1933)) leads to the mean value theorem: there exists z such that

$$\sum_{i=1}^{n} \frac{f(x_i)}{\prod_{j\neq i}(g(x_i)-g(x_j))} = \frac{\Delta^{n-1}(z)}{(n-1)!} \tag{3}$$

with $\min(x_1,\dots,x_n) \leq z \leq \max(x_1,\dots,x_n)$ and where

$$\Delta^{(1)} = \frac{f'}{g'},\dots,\Delta^{(n-1)} = \frac{(\Delta^{(n-2)})'}{g'}.$$

Choosing $f(x) = x^r$ and $g(x) = x^s$ and solving for z again, results in the generalized Stolarsky mean:

$$N_{r,s}(x_1,\dots,x_n) = \left[\frac{(n-1)!\ \ s^{n-1}}{r(r-s)(r-2s)\dots(r-(n-2)s)} \sum_{i=1}^{n} \frac{x_i^r}{\prod_{j\neq i}(x_i^s - x_j^s)}\right]^{\frac{1}{r-(n-1)s}} \tag{4}$$

$$(r,s \neq 0;\ r/s \neq 1,\dots,(n-1)).$$

Clearly, at $n = 2$, (4) and (2) are identical. Furthermore, $N_{r,s}(x_1,\dots,x_n)$ includes the power mean $M_s = \left(\frac{1}{n}\sum x_i^s\right)^{1/s}$ at $r = ns$. This result follows from

$$\sum \frac{x_i^{ns}}{\prod(x_i^s - x_j^s)} = \sum x_i^s.$$

So the question is, why has the power mean M_s occupied so much of the attention of the profession, whereas the more general $N_{r,s}$ and similar expressions (see Leach and Scholander (1984)) seem to have gone more or less unnoticed.

3 On a specific mean value property postulated by Kolmogoroff and Nagumo

The reason for putting emphasis on power means seems to be hidden in two papers that were published more than sixty years ago. Kolmogoroff (1930) and Nagumo (1930) simultaneously and independently worked out the axioms required to single out the power mean M_s. Apart from 'natural' properties such as symmetry, homogeneity, boundedness ($\min(x_i) \leq M \leq \max(x_i)$) and monotonicity, they demanded that a mean M should comply with the following characteristic:

$$M_n(x_1,\dots,x_n) = M_n(M_r(x_1,\dots,x_r) \circ r, x_{r+1},\dots,x_n) \tag{5}$$

where $M_r(x_1,\dots,x_r) \circ r$ stands for "r times the argument $M_r(x_1,\dots,x_r)$". Kolmogoroff and Nagumo thus demonstrated that it is property (5) by which the power mean M_s is distinguished from other means. We have to conclude, therefore, that $N_{r,s}(x_1,\ldots,x_n)$ from (4) is a deficient mean value function in the sense that it lacks property (5) (except at $r = ns$).

The most serious consequence is this: in (3) and consequently in (4) we have (implicitly) assumed the x_i's to be different. With some x_i's being equal instead, the appropriate number of divided differences in (3) has to be replaced by derivatives. But the result of (4), then, depends on the sequence of differentiation. Specificially, $N_{r,s}(x,x,y,y)$ say, is different from $N_{r,s}(x,y)$ and is not even uniquely determined, because $N_{r,s}(x_1,\ldots,x_n)$ is not continous at $x_i = x_j$, except at $r = ns$.

To put this consequence in slightly different terms: it is not possible to have a meaningful definition of 'weighted' means if property (5) is being neglected.

But in the case of two arguments only, property (5) is meaningless. In fact, the technique of proof in Kolmogoroff (1930) and Nagumo (1930) breaks down at $n = 2$. Thus it is indeed unnecessarily restrictive in the case of two arguments to confine attention to power means. Specificially, $N_{r,s}(x,y)$ from (2) is deficient in no respect, as it fulfills all the Kolmogoroff-Nagumo axioms. However, generalizing these Stolarsky means (or other means) to be able to handle more than two arguments can be done only, if one is willing to forgo property (5) and suffer the consequences.

4 A mean value function in two arguments that ought to be more popular

At $r = s+1$ the mean $N_{r,s}(x,y)$ reduces to

$$\begin{aligned} P_s(x,y) &= \frac{s(x^{s+1}-y^{s+1})}{(s+1)(x^s-y^s)} && s \neq -1,0 \\ P_{-1}(x,y) &= xy\frac{\ln x - \ln y}{x-y} && (s=-1) \\ P_0(x,y) &= \frac{x-y}{\ln x - \ln y} && (s=0) \end{aligned} \tag{6}$$

$P_0(x,y)$ has won some popularity in recent years under the name 'logarithmic mean' (Carlson 1972). We concentrate on this logarithmic mean in section 5 and shall write $P_0(x,y) = L(x,y)$ here-after.

The following scheme concentrates on the 5 most relevant elements of $P_s(x,y)$ where H, G, L and A denote the harmonic, geometric, logarithmic, and arithmetic mean, respectively.

s	$P_s(x,y)$	$P_s(x,x^{-1})$
-2	$H = G^2/A$	$1/A$
-1	G^2/L	$1/L$
$-\frac{1}{2}$	G	1
0	L	L
1	A	A

We conclude that $P_s(x,y)$ resembles the traditional power mean $M_s(x,y)$ in bridging the gap from H to G to A and that $P_s(x,y)$ has the additional advantage of including as well the logarithmic mean L (and the still unbaptized G^2/L).

Stolarsky (1975) proves that $N_{r,s}(x,y)$ increases monotonically in both the parameters r and s. It follows that $P_s(x,y)$ increases monotonically in s which leads to the inequality

$$H \leq G^2/L \leq G \leq L \leq A$$

through which we are informed that the logarithmic mean is bounded numerically by the geometric mean and the arithmetic mean.

To prove our point that $P_s(x,y)$ is a more interesting mean value function than the traditional power mean $M_s(x,y)$ (as long as we concentrate on two arguments only), we have to demonstrate that the logarithmic mean is useful in applications.

5 Applications of the logarithmic mean

Burkhardt (1933) seems to have been the first to give a direct application of the logarithmic mean. Assume the population size $B(t)$ of birth-cohort to change exponentially:

$$B(t) = ae^{\gamma t}$$

and suppose that the total amount of person-years within a given period $[t, t+1]$ (one of the basic concepts of life-table construction) is the problem under investigation.

Then we have

$$\int_t^{t+1} B(x)dx = L(B(t+1), B(t))$$

because of $\gamma = \ln B(t+1) - \ln B(t)$. To calculate the amount of person-years it is sufficient, therefore, to have census results at the beginning and at the end of the period with the logarithmic mean doing the trick.

Another application of $L(x,y)$ is to be found in Leach and Scholander (1978): It is well known that the following inequality holds:

$$\left(1+\frac{1}{n}\right)^n \leq e \leq \left(1+\frac{1}{n}\right)^{n+1}.$$

So the question arises, if there is a mean $M(n, n+1)$ for which

$$\left(1+\frac{1}{n}\right)^{M(n,n+1)} = e$$

is true. The answer is in the affirmative: choose $M(n,n+1) = L(n,n+1)$. The resulting identity can easily be extended to

$$\left(1+\frac{x}{n}\right)^{L(n,n+x)} = e^x.$$

Another identity involving the logarithmic mean, which has some relevance to claim (see Lorenzen 1989), is

$$\tanh^{-1}(x) = \frac{x}{L(1+x, 1-x)}$$

which contrasts with $x = x/A(1+x, 1-x)$ so that

$$f_s(x) = \frac{x}{P_s(1+x, 1-x)}$$

includes $f_1(x) = x$ as well as $f_0(x) = \tanh^{-1}(x)$.

More applications of the logarithmic mean (and other non-standard means) are to be found in Vartia (1976) and Lorenzen (1990, 1992).

6 Conclusion

Whereas it is usually well justified to concentrate on the traditional family of power means, as only these means comply with the Kolmogoroff-Nagumo-property (5), it is unnecessarily restrictive to do so in the case of means in two arguments. Non-standard means in two arguments may have all the required properties and have some relevance to claim within applications.

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