Partitioning Balls into Topologically Equivalent Pieces

▌Elemente der Mathematik

Partitioning Balls into Topologically Equivalent Pieces

Christian Richter

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thematik an der Friedrich-Schiller-Universität in Jena. Gegenwärtig ist er wissen-
schaftlicher Assistent am dortigen mathematischen Institut und beschäftigt sich in
Vorbereitung auf eine Habilitation mit Zerlegungs- und Überdeckungsproblemen me-
trischer Räume und mit approximationstheoretischen Anwendungen. In seiner Frei-
zeit unternimmt er gerne Wanderungen und findet Entspannung beim Musikhören.

1 Introduction

In 1949 B. L. van der Waerden asked for a proof for the non-existence of a disjoint
decomposition $B = A_1 \cup A_2$ of a closed ball (circle) in the Euclidean plane $\mathbb{E}^2$ into
two congruent pieces A_1 and A_2 (cf. [Wae]). This problem found, beside its solution in
the same issue of "Elemente der Mathematik", many generalizations (cf. [Wa1], [Wa2],
[He], [Ede1], [Ede2], [E/J/T], [Ri]). We use the most general notation from [Wa1]: Let
$\mathcal{G}$ be a group acting on a set X. A subset $Y \subseteq X$ is called *n-divisible w.r.t.* $\mathcal{G}$ if Y may
be partitioned into n pieces which are pairwise congruent via $\mathcal{G}$.

Im Jahre 1949 veröffentlichten die *Elemente der Mathematik* eine Aufgabe von van
der Waerden:

*Ist es möglich, eine Kreisscheibe in zwei zueinander fremde kongruente Punktmengen zu
zerlegen? (Ob man den Randkreis zur Kreisscheibe rechnet oder nicht, ist gleichgültig.
Das einfachste ist, den Randkreis dazu zu rechnen.)*

In den vergangenen 50 Jahren wurden verschiedene Verallgemeinerungen dieses Pro-
blems behandelt. Sie betreffen insbesondere den Fall höherer Dimensionen und den
Fall, wo statt kongruenter Teile topologisch äquivalente Teile betrachtet werden. –
Christian Richter beweist im vorliegenden Beitrag das folgende allgemeine Resultat:

*Eine abgeschlossene Kugel B im d-dimensionalen euklidischen Raum lässt genau dann
eine Zerlegung in n topologisch äquivalente Teile zu, wenn das Paar (d,n) vom Paar
(1,2) verschieden ist. ust*

In the Euclidean space $\mathbb{E}^d$ it is the natural problem to discuss the n-divisibility of sets w.r.t. the group $\mathscr{B}^d$ of all isometries of $\mathbb{E}^d$. In the paper [He] it is shown that every closed, bounded and convex set in $\mathbb{E}^d$ is not 2-divisible w.r.t. $\mathscr{B}^d$. Moreover it is proved that closed intervals are not n-divisible w.r.t. $\mathscr{B}^1$ for all $n \in \{2, 3, 4, \ldots\}$. E. Hertel asks for the largest group of transformations of $\mathbb{E}^d$ for which these results can be generalized.

Of course, half-open intervals $[a, b) \subseteq \mathbb{E}^1$ are n-divisible w.r.t. translations. Similarly, the small manipulation of removing its center makes a circle in the Euclidean plane n-divisible w.r.t. rotations. This shows that "small" topological changes can change the property of n-divisibility of sets. On that account it becomes interesting to consider the n-divisibility of bounded convex sets w.r.t. the group $\mathscr{T}^d$ of all homeomorphisms of $\mathbb{E}^d$ onto itself. It is obviously sufficient to restrict the considerations to balls, which represent the topological type of bounded convex sets. We call a set $B \subseteq \mathbb{E}^d$ a *(general) ball* of radius $r > 0$ centered in $x_0 \in \mathbb{E}^d$ if

$$\left\{x \in \mathbb{E}^d : \|x - x_0\| < r\right\} \subseteq B \subseteq \left\{x \in \mathbb{E}^d : \|x - x_0\| \leq r\right\} .$$

2 Partitioning closed balls

Before we discuss the n-divisibility of closed balls in $\mathbb{E}^d$, we ask for partitions into n pairwise homeomorphic topological subspaces. This property of the partition sets is more general than the pairwise congruence via $\mathscr{T}^d$, since a homeomorphic bijection between two subsets of $\mathbb{E}^d$ can not be extended to a transformation from $\mathscr{T}^d$ in general.

Theorem 1 *Let B be a closed ball in $\mathbb{E}^d$ ($d \geq 1$) and let $n \geq 2$. Then there exists a disjoint decomposition of B into n pairwise homeomorphic topological subspaces of $\mathbb{E}^d$.*

Proof. The cases $(d, n) \neq (1, 2)$ will be discussed in the proof of the following Theorem 2.
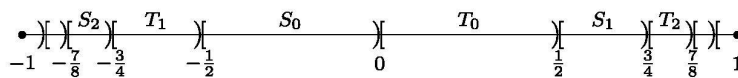
Fig. 1

If $d = 1$ and $n = 2$ we assume that $B = [-1, 1]$ without loss of generality (cf. Fig. 1). We consider the decomposition $B = S \cup T$ where

$$S = \{-1\} \cup \bigcup_{k=0}^{\infty} S_k \quad \text{with} \quad S_k = \begin{cases} [-1 + 2^{-(k+1)}, -1 + 2^{-k}) & \text{if } k \text{ is even,} \\ [1 - 2^{-k}, 1 - 2^{-(k+1)}) & \text{if } k \text{ is odd} \end{cases}$$

and

$$T = \{1\} \cup \bigcup_{k=0}^{\infty} T_k \quad \text{with} \quad T_k = \begin{cases} [1 - 2^{-k}, 1 - 2^{-(k+1)}) & \text{if } k \text{ is even,} \\ [-1 + 2^{-(k+1)}, -1 + 2^{-k}) & \text{if } k \text{ is odd.} \end{cases}$$

A homeomorphism φ from S onto T can be obtained as follows: We define the restriction $\varphi|_{S_k}$ of φ to the interval S_k, $k \in \{0, 1, 2, \ldots\}$, to be the translation of S_k onto T_k and, of course, $\varphi(-1) = 1$. The continuity of φ in a point $x_0 \in S \setminus \{-1\}$, say $x_0 \in S_{k_0}$, is implied by the existence of a neighbourhood $U(x_0)$ in the topological space S such that $U(x_0) \subseteq S_{k_0}$. Hence $\varphi|_{U(x_0)}$ is a restriction of a translation and, consequently, φ is continuous in x_0. If $x_0 = -1$ we observe, that all points $x \in S$ with $\|x - x_0\| \leq 2^{-2i}$, i.e. $x \in \{-1\} \cup \bigcup_{l=i}^{\infty} S_{2l}$, are mapped into $\{1\} \cup \bigcup_{l=i}^{\infty} T_{2l}$, i.e. $\|\varphi(x) - \varphi(x_0)\| = \|\varphi(x) - 1\| \leq 2^{-2i}$.

The continuity of φ^{-1} can analogously be checked. This completes the proof. $\square$

The investigation of the n-divisibility w.r.t. $\mathcal{T}^d$ requires several steps. The first proposition shows that this n-divisibility is strictly harder than the partition property considered in Theorem 1. (Proposition 1 is proved in a more general context in [Ri]. In the present paper we give an elementary proof.)

Proposition 1 *Closed balls of $\mathbb{E}^1$ are not 2-divisible w.r.t. $\mathcal{T}^1$.*

Proof. Without loss of generality we consider the ball $B = [0, 1]$. We assume the contrary, i.e. there exist two disjoint sets S and T and a homeomorphism $\tau \in \mathcal{T}^1$ such that

$$[0, 1] = S \cup T, \quad \tau(S) = T \quad \text{and} \quad 0 \in S. \tag{1}$$

Fact 1: *τ has no fixed point $x_0 = \tau(x_0)$ in $[0, 1]$.*
Such a point would belong to both the sets S and T, since $\tau(S) = T$, in contradiction to the disjointness of S and T.

Fact 2: *τ is either strictly increasing or strictly decreasing.*
This is simply implied by the injectivity of τ and Cauchy's intermediate value theorem.

Case 1: *τ is strictly increasing.*
We prove the following statement by induction w.r.t. i:

Fact 3: *Let $i \in \{0, 1, 2, \ldots\}$. Then*

(α_i) $\qquad\qquad \tau^{2i}(0) < \tau^{2i+1}(0) < \tau^{2i+2}(0)$,

(β_i) $\qquad\qquad [\tau^{2i+2}(0), 1] = (S \cap [\tau^{2i+2}(0), 1]) \cup (T \cap [\tau^{2i+2}(0), 1])$,

(γ_i) $\qquad\qquad \tau(S \cap [\tau^{2i+2}(0), 1]) = T \cap [\tau^{2i+2}(0), 1]$ $\qquad$ *and*

(δ_i) $\qquad\qquad \tau^{2i+2}(0) \in S \cap [\tau^{2i+2}(0), 1]$.

We start with $i = 0$. The first inequality from (α_0) is easily seen, since

$$\tau^1(0) \in \tau(S) = T = [0, 1] \setminus S \subseteq (0, 1]$$

and thus $\tau^0(0) = 0 < \tau^1(0)$. Application of τ to this inequality yields $\tau^1(0) < \tau^2(0)$, which completes the verification of (α_0). Equation (β_0) is trivial in accordance with (1). Moreover we have

$$\min(T) = \min(\tau(S)) = \tau(\min(S)) = \tau(0),$$

hence
$$[0, \tau(0)) \subseteq S$$
and, by application of τ,
$$\left[\tau(0), \tau^2(0)\right) \subseteq T .$$

These inclusions imply that
$$S \cap \left[0, \tau^2(0)\right) = [0, \tau(0)) \quad \text{and} \quad T \cap \left[0, \tau^2(0)\right) = \left[\tau(0), \tau^2(0)\right) . \qquad (2)$$

We obtain (γ_0) by
$$\begin{aligned}
\tau\left(S \cap \left[\tau^2(0), 1\right]\right) &= \tau\left(S \setminus \left[0, \tau^2(0)\right)\right) \\
&= \tau\left(S \setminus [0, \tau(0))\right) && \text{(by (2))} \\
&= T \setminus \left[\tau(0), \tau^2(0)\right) \\
&= T \setminus \left[0, \tau^2(0)\right) && \text{(by (2))} \\
&= T \cap \left[\tau^2(0), 1\right] .
\end{aligned}$$

Finally, we have $\tau^2(0) \in [0, 1]$, since $\tau^2(0)$ belongs to the closure of T in accordance with the second part of (2). If $\tau^2(0)$ would belong to T we would obtain
$$\tau(0) = \tau^{-1}\left(\tau^2(0)\right) \in \tau^{-1}(T) = S$$
in contradiction to (2). This shows the inclusion $\tau^2(0) \in S$ and completes the verification of (δ_0).

The step from $i - 1$ to i, $i \geq 1$, can similarly be done. We presuppose (β_{i-1}), (γ_{i-1}), and (δ_{i-1}) instead of (1), i.e. we replace $[0, 1]$ by $\left[\tau^{2i}(0), 1\right]$, S by $S \cap \left[\tau^{2i}(0), 1\right]$, T by $T \cap \left[\tau^{2i}(0), 1\right]$, and 0 by $\tau^{2i}(0)$. Then we infer (α_i), (β_i), (γ_i), and (δ_i) by the same arguments as above. This completes the proof of Fact 3.
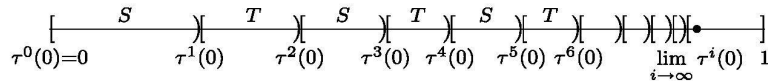

Fig. 2

According to Fact 3 we have an increasing sequence $\left(\tau^i(0)\right)_{i=0}^{\infty}$ in $[0, 1]$ (cf. Fig. 2). The limit $x_0 \in [0, 1]$ of this sequence is a fixed point of τ, since
$$\tau(x_0) = \tau\left(\lim_{i \to \infty} \tau^i(0)\right) = \lim_{i \to \infty} \tau^{i+1}(0) = x_0 .$$

Hence we obtained a contradiction to Fact 1. This closes the considerations of the first case.

Case 2: τ *is strictly decreasing.*
We have $\tau(0) \in \tau(S) \subseteq [0, 1]$. According to Fact 1 we obtain $0 < \tau(0) \leq 1$. Moreover we have $\tau(1) < \tau(0) \leq 1$. Consequently, $\tau(0) - \text{id}(0) > 0$ and $\tau(1) - \text{id}(1) < 0$. Hence there exists an $x_0 \in (0, 1)$ with $\tau(x_0) - x_0 = 0$ in contradiction to Fact 1. This completes the proof of Proposition 1. $\square$

Proposition 2 *Every closed ball in* $\mathbb{E}^1$ *is n-divisible w.r.t.* $\mathcal{T}^1$ *for all integers* $n \in \{3, 4, 5, \ldots\}$.

Proof. We consider the ball $B = [0, n]$, which can be decomposed into the n intervals $[0, 1), [1, 2), \ldots, [n-2, n-1), [n-1, n]$ (cf. Fig. 3). The half-open intervals $[j-1, j)$, $1 \le j \le n-1$, possess a partition

$$[j - 1, j) = S_j \cup T_j \qquad \text{with}$$

$$S_j = \left\{ j - \tfrac{1}{2} \right\} \cup \bigcup_{k=1}^{\infty} \left(\left\{ j - \tfrac{1}{2} \right\} + (-1)^k \cdot \left[2^{-k-1}, 2^{-k} \right] \right) ,$$

$$T_j = \bigcup_{k=1}^{\infty} \left(\left\{ j - \tfrac{1}{2} \right\} + (-1)^{k+1} \cdot \left(2^{-k-1}, 2^{-k} \right) \right) .$$

The remaining interval $[n - 1, n]$ can be decomposed into

$$[n - 1, n] = S_n \cup T_n \qquad \text{with}$$

$$S_n = \left\{ n - \tfrac{1}{2} \right\} \cup \bigcup_{k=1}^{\infty} \left(\left\{ n - \tfrac{1}{2} \right\} + \left[-2^{-2k+1}, -2^{-2k} \right] \cup \left[2^{-2k}, 2^{-2k+1} \right] \right) ,$$

$$T_n = \bigcup_{k=1}^{\infty} \left(\left\{ n - \tfrac{1}{2} \right\} + \left(-2^{-2k}, -2^{-2k-1} \right) \cup \left(2^{-2k-1}, 2^{-2k} \right) \right) .$$

We consider the following partition:

$$B = A_1 \cup A_2 \cup \ldots \cup A_n \qquad \text{with}$$

$$A_j = \begin{cases} S_j \cup T_{j+1} & \text{if } 1 \le j \le n - 1 , \\ S_n \cup T_1 & \text{if } j = n . \end{cases}$$

All the sets $A_1, A_2, \ldots, A_{n-1}$ and the reflection $-A_n$ of A_n have the following topological structure:

$$A = \left(\bigcup_{i=1}^{\infty} \hat{C}_i \right) \cup \{c\} \cup \left(\bigcup_{i=1}^{\infty} \check{C}_i \right) \cup \left(\bigcup_{i=1}^{\infty} \hat{O}_i \right) \cup \left(\bigcup_{i=1}^{\infty} \check{O}_i \right) .$$

The subsets $\hat{C}_i$ form an increasing sequence of closed and bounded intervals of positive length which tends to a point $c \in A$. $(\check{C}_i)_{i=1}^{\infty}$ is a decreasing sequence of closed and bounded intervals of positive length tending to c from above. Similarly, the sets $\hat{O}_i$ are open and bounded intervals of positive length which accumulate in $o \notin A$ from below. The sequence $(\check{O}_i)_{i=1}^{\infty}$ consists of open and bounded intervals of positive length and tends to o from above. All the intervals $\hat{C}_i$, $\check{C}_i$, $\hat{O}_i$, and $\check{O}_i$ are pairwise separated by intervals of positive length. Moreover the closed set consisting of $\{c\}$ and the closed intervals is separated by an interval from the open set formed by the open intervals, too.
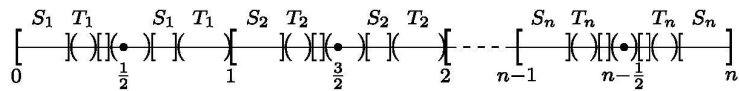
Fig. 3

For any two given sets A and A' of that type there exists a strictly increasing homeomorphism $\tau \in \mathcal{T}^1$ such that $\tau(A) = A'$. One can define τ piecewise linearly where $\hat{C}_1$ is mapped onto $\hat{C}_1'$, $\hat{C}_2$ onto $\hat{C}_2'$, ..., c onto c', ..., $\check{C}_2$ onto $\check{C}_2'$, $\check{C}_1$ onto $\check{C}_1'$, $\hat{O}_1$ onto $\hat{O}_1'$, $\hat{O}_2$ onto $\hat{O}_2'$, ..., o onto o', ..., $\check{O}_2$ onto $\check{O}_2'$, and $\check{O}_1$ onto $\check{O}_1'$. This shows, that the sets A_i, $1 \le i \le n$, are pairwise congruent via $\mathcal{T}^1$, and completes the proof of Proposition 2. □

Let us remark that the above construction does not apply to the case $n = 2$, since the two sets S_2 and T_1 would not be separated by an interval. This would change the topological structure of $A_2 = S_2 \cup T_1$.

Proposition 3 *Every closed ball in $\mathbb{E}^2$ is 2-divisible w.r.t. $\mathcal{T}^2$.*

Proof. We consider the rhomb $B = \left\{ (x,y) \in \mathbb{E}^2 : |x| \le 1, |y| \le \frac{1}{4}(1 - |x|) \right\}$ instead of a ball (cf. Fig. 4). B admits the decomposition $B = S \cup T$ where

$$S = \{(-1,0)\} \cup \bigcup_{k=0}^{\infty} S_k \qquad \text{and} \qquad T = \{(1,0)\} \cup \bigcup_{k=0}^{\infty} T_k \qquad \text{with}$$

$$S_k = \begin{cases} \left\{ (x,y) \in B : x \in \left[-1 + 2^{-(k+1)}, -1 + 2^{-k}\right) \right\} & \text{if } k \text{ is even,} \\ \left\{ (x,y) \in B : x \in \left[1 - 2^{-k}, 1 - 2^{-(k+1)}\right) \right\} & \text{if } k \text{ is odd,} \end{cases} \qquad \text{and}$$

$$T_k = \begin{cases} \left\{ (x,y) \in B : x \in \left[1 - 2^{-k}, 1 - 2^{-(k+1)}\right) \right\} & \text{if } k \text{ is even,} \\ \left\{ (x,y) \in B : x \in \left[-1 + 2^{-(k+1)}, -1 + 2^{-k}\right) \right\} & \text{if } k \text{ is odd.} \end{cases}$$

(This partition is closely related to that given in the proof of Theorem 1.)

We define a homeomorphism $\tau \in \mathcal{T}^2$ as a product $\tau = \varphi \circ \sigma$ where σ is the reflection on the vertical axis, i.e. $\sigma(x,y) = (-x,y)$. The structure of φ is more difficult: For all the sets $\sigma(S_k)$, which are of the form $\sigma(S_k) = \{(x,y) \in B : x \in (a_k, b_k]\}$, we consider a covering square $R_k = \left[a_k - \frac{b_k - a_k}{3}, b_k + \frac{b_k - a_k}{3} \right] \times \left[-\frac{5(b_k - a_k)}{6}, \frac{5(b_k - a_k)}{6} \right]$. The closed squares R_k are mutually separated. The restriction $\varphi|_{R_k}$ to R_k is a homeomorphism of R_k onto itself which leaves the boundary $\mathrm{bd}(R_k)$ pointwise fixed and maps $\sigma(S_k)$ onto T_k. Besides that, we demand the remaining points $(x,y) \notin \bigcup_{k=0}^{\infty} R_k$ to be fixed under φ. It is easily seen that the piecewise definitions of φ fit together and form a homeomorphism $\varphi \in \mathcal{T}^2$.
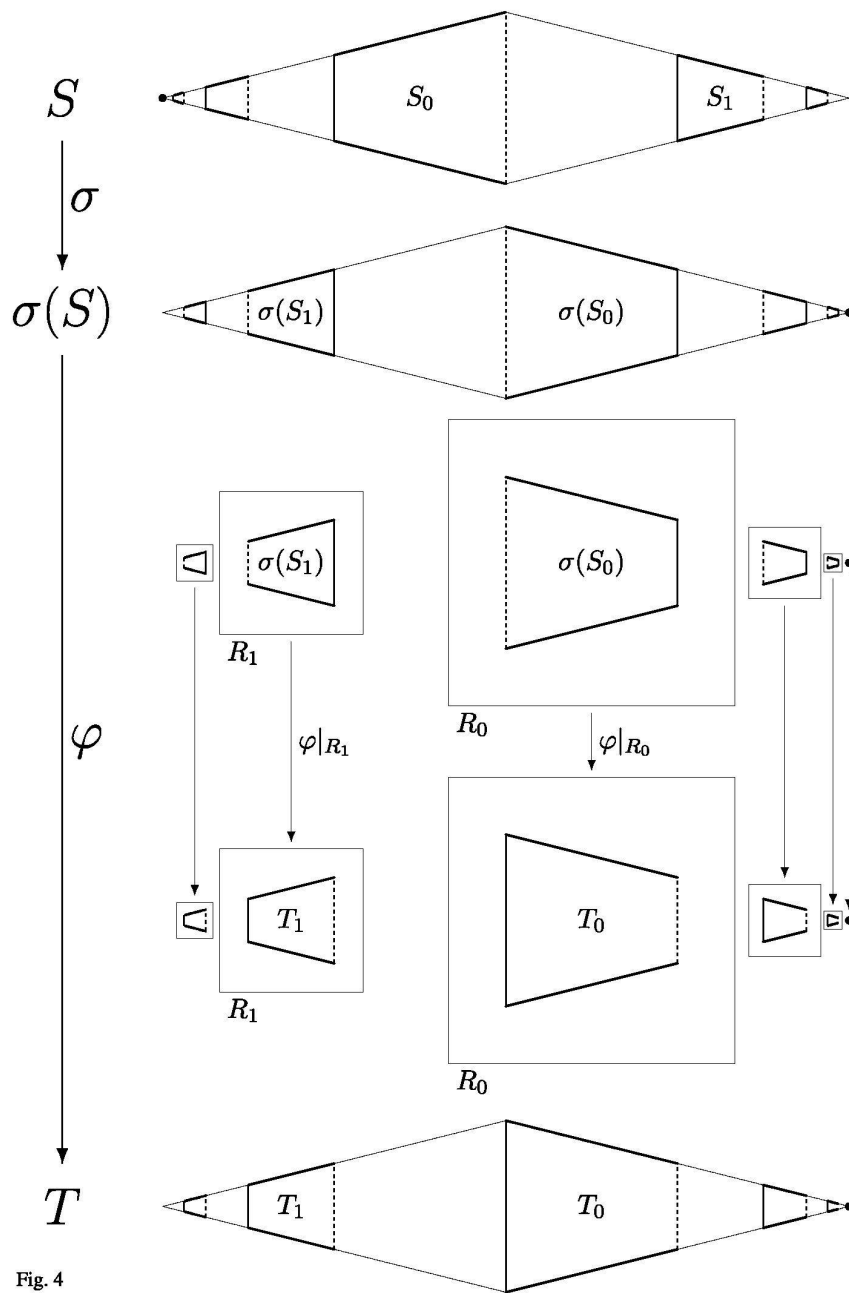
Fig. 4

Consequently, we obtain $\tau = \varphi \circ \sigma \in \mathcal{T}^2$ such that

$$\tau(S) = \varphi \circ \sigma \left(\{(-1,0)\} \cup \bigcup_{k=0}^{\infty} S_k \right)$$

$$= \varphi\big(\sigma(\{(-1,0)\})\big) \cup \bigcup_{k=0}^{\infty} \varphi\big(\sigma\left(S_k\right)\big)$$

$$= \{(1,0)\} \cup \bigcup_{k=0}^{\infty} T_k \quad = \quad T \, .$$

This completes the verification of Proposition 3.   □

Proposition 4 *Let* $d \in \{2,3,4,\ldots\}$ *and* $n \in \{2,3,4,\ldots\}$ *such that a closed ball in* $\mathbb{E}^{d-1}$ *is* n-*divisible w.r.t.* $\mathcal{T}^{d-1}$. *Then every closed ball in* $\mathbb{E}^d$ *is* n-*divisible w.r.t.* $\mathcal{T}^d$.

Proof. According to the assumption there exist a closed ball $B^{d-1} \in \mathbb{E}^{d-1}$, a decomposition $B^{d-1} = S_1^{d-1} \cup S_2^{d-1} \cup \ldots \cup S_n^{d-1}$ of B^{d-1}, and mappings τ_i^{d-1} from $\mathcal{T}^{d-1}$ such that $\tau_i^{d-1}\left(S_1^{d-1}\right) = S_i^{d-1}$, $1 \le i \le n$. Let B^d be a closed ball in $\mathbb{E}^d$. We assume that $B^d = B^{d-1} \times [0,1]$ without loss of generality. Of course, B^d admits the decomposition $B^d = \left(S_1^{d-1} \times [0,1]\right) \cup \left(S_2^{d-1} \times [0,1]\right) \cup \ldots \cup \left(S_n^{d-1} \times [0,1]\right)$. Moreover, if the identity on $\mathbb{E}^1$ is denoted by ι, then the mappings $\tau_i^d = \tau_i^{d-1} \times \iota : \mathbb{E}^d \to \mathbb{E}^d$, $1 \le i \le n$, belong to $\mathcal{T}^d$ and fulfil $\tau_i^d\left(S_1^{d-1} \times [0,1]\right) = S_i^{d-1} \times [0,1]$. This shows the n-divisibility of B^d w.r.t. $\mathcal{T}^d$.   □

Propositions 1–4 amount to the following theorem:

Theorem 2 *Let* $d \in \{1,2,3,\ldots\}$, $n \in \{2,3,4,\ldots\}$, *and let* B *be a closed ball in* $\mathbb{E}^d$. *Then* B *is* n-*divisible w.r.t.* $\mathcal{T}^d$ *if and only if* $(d,n) \ne (1,2)$.   □

3 Partitioning non-closed balls

Proposition 5 *Let* S *be a subset of* $\mathbb{E}^d$, $x_0 \in \mathrm{bd}(S) \setminus S$ *a point such that* $S \cup \{x_0\}$ *is star-shaped w.r.t.* x_0, $n \in \{2,3,4,\ldots\}$, *and* δ *the dilatation with center* x_0 *and factor* $c > 1$. *Then there exists a decomposition*

$$S = S_1 \cup S_2 \cup \ldots \cup S_n$$

of S *such that*

$$S_i = \delta^{i-1}\left(S_1\right) \qquad for \qquad i = 1,2,\ldots,n \, .$$

Proof. It is sufficient to show the assertion for all intersections

$$S^H = S \cap H$$

of S with open halflines H starting in x_0, since δ maps H onto itself. We assume that $S^H \subseteq \mathbb{E}^1$, $x_0 = 0$, and $H = (0,\infty)$ without loss of generality. S^H is a bounded or an

unbounded interval, i.e. $S^H = \emptyset$, $S^H = (0, b)$, $S^H = (0, b]$, or $S^H = (0, \infty)$. If S^H is empty then there is nothing to show. Otherwise we consider the decomposition

$$S^H = S_1^H \cup S_2^H \cup \ldots \cup S_n^H$$

with

$$S_i^H = \begin{cases} \displaystyle\bigcup_{k=-\infty}^{-1} \left[b \cdot c^{nk+i-1}, b \cdot c^{nk+i} \right) & \text{if } S^H = (0, b), \\[2ex] \displaystyle\bigcup_{k=-\infty}^{-1} \left(b \cdot c^{nk+i-1}, b \cdot c^{nk+i} \right] & \text{if } S^H = (0, b], \\[2ex] \displaystyle\bigcup_{k=-\infty}^{\infty} \left(c^{nk+i-1}, c^{nk+i} \right] & \text{if } S^H = (0, \infty). \end{cases}$$

A simple calculation shows that

$$S_i^H = c^{i-1} \cdot S_1^H = \delta^{i-1}\left(S_1^H\right)$$

for $i = 1, 2, \ldots, n$ and completes the proof. $\square$

Of course, for every strictly convex set $S \subseteq \mathbb{E}^d$, which is not closed, there exists a point $x_0 \in \mathrm{bd}(S) \setminus S$ as assumed in Proposition 5. Hence we obtain:

Corollary *Every non-closed strictly convex set in $\mathbb{E}^d$ is n-divisible w.r.t. the group $\mathcal{H}^d$ of all similarities of $\mathbb{E}^d$ for $n = 2, 3, 4, \ldots$.* $\square$

In the context of the present paper we formulate the following more special implication concerning the divisibility of non-closed balls w.r.t. $\mathcal{T}^d$:

Theorem 3 *Let $d \in \{1, 2, 3, \ldots\}$, $n \in \{2, 3, 4, \ldots\}$, and let B be a non-closed ball in $\mathbb{E}^d$. Then B is n-divisible w.r.t. $\mathcal{T}^d$.* $\square$

Let us close this paper with a remark on the structure of the decompositions of closed or open balls in $\mathbb{E}^1$ considered in the proofs of Proposition 2 and Proposition 5. The balls are partitioned into subsets mutually congruent via $\mathcal{T}^1$, any of the subsets consisting of countably many connected components. The partitions given in this paper are optimal in the following sense:

Proposition 6 *Let $B = S_1 \cup S_2 \cup \ldots \cup S_n$ be a decomposition of a closed or open ball $B \in \mathbb{E}^1$ into $n \geq 2$ subsets which are pairwise congruent via $\mathcal{T}^1$. Then all the sets S_i consist of infinitely many connected components.*

Proof. We assume the contrary, i.e. the boundaries $\mathrm{bd}(S_i)$ of the subsets S_i w.r.t. the topology of $\mathbb{E}^1$ are finite. Then the interiors of the sets S_i consist of k open intervals, where $k \geq 1$ does not depend on i according to the congruence of the sets S_i via $\mathcal{T}^1$. Hence the ball B is decomposed into nk open intervals – the components of $\mathrm{int}(S_i)$, $1 \leq i \leq n$, – and those points of the boundaries of the sets S_i which belong to B. The nk intervals are separated by $nk - 1$ inner points of B. Consequently,

$$\mathrm{card}\left(\bigcup_{i=1}^{n} \left(\mathrm{bd}(S_i) \cap B \right) \right) = \begin{cases} nk + 1 & \text{if } B \text{ is a closed interval}, \\ nk - 1 & \text{if } B \text{ is an open interval}. \end{cases} \tag{3}$$

On the other hand we have

$$\bigcup_{i=1}^{n}\big(\mathbf{bd}(S_i)\cap B\big) = \bigcup_{i=1}^{n}\big(\mathbf{bd}(S_i)\cap S_i\big)\ ,$$

where the right union is disjoint. All the sets $\mathbf{bd}(S_i)\cap S_i$, $1 \le i \le n$, are of the same cardinality l, since the sets S_i are congruent via $\mathcal{T}^1$. Thus we obtain

$$\mathrm{card}\left(\bigcup_{i=1}^{n}\big(\mathbf{bd}(S_i)\cap B\big)\right) = nl\ .$$

This contradiction to formula (3) proves Proposition 6.   □

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References

[Ede1]   M. Edelstein: *Isometric decompositions*, J. London Math. Soc. **37** (1988), 158–163

[Ede2]   M. Edelstein: *On the indecomposability of compact convex sets*, Proc. Amer. Math. Soc. **115** (1992), 737–739

[E/J/T] M. Edelstein, K. Johnson, A. C. Thompson: *On the isometric dissection problem for convex sets*, Studia Sci. Math. Hungar. **27** (1992), 273–277

[He]   E. Hertel: *Disjunkte Pflasterungen konvexer Körper*, Studia Sci. Math. Hungar. **21** (1986), 379–386

[Ri]   C. Richter: *On decompositions of compact convex sets*, Geom. Dedicata **71** (1998), 1–4

[Wae]   B. L. van der Waerden: *Aufgabe 51*, Elem. Math. **4** (1949), 18

[Wa1]   S. Wagon: *Partitioning intervals, spheres and balls into congruent pieces*, Canad. Math. Bull. **26** (3) (1983), 337–340

[Wa2]   S. Wagon: *The Banach-Tarski paradox*, Cambridge Univ. Press, 1986

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