Three points related to the incenter and excenters of a triangle

Elem. Math. 61 (2006) 74 – 80
0013-6018/06/020074-7

Elemente der Mathematik

Three points related to the incenter and excenters of a triangle

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1 Introduction

Let $\Delta := \{A, B, C\}$ be a triangle in the Euclidean plane. The side lengths of Δ shall be denoted by $c := \overline{AB}$, $b := \overline{AC}$ and $a := \overline{BC}$. The interior angles enclosed by the edges of Δ are $\beta := \angle ABC$, $\gamma := \angle BCA$ and $\alpha := \angle CAB$, see Fig. 1.

It is well-known that the bisectors w_α, w_β and w_γ of the interior angles of Δ are concurrent in the *incenter* I of Δ. The bisectors $\overline{w_\beta}$ and $\overline{w_\gamma}$ of the exterior angles at the vertices A and B and w_α are concurrent in the center A_1 of the *excircle* touching Δ along BC from the outside.

Changing α, β and γ cyclically, we can find the remaining two *excenters* A_2 and A_3. To get familiar with the notations used in this paper, see Fig. 1.

Here we remark that the base triangle Δ is the *orthoptic triangle* of the triangle built by the excenters. The orthocenter of Δ is the incenter of the orthoptic triangle. Later, when we give our theorems, a second interpretation will use this fact.

Die Inkreismitte und die drei Ankreismitten eines Dreiecks Δ sind die Mittelpunkte jener vier Kreise, die alle Seiten von Δ berühren. Aus diesen Punkten lassen sich drei weitere Punkte ableiten. Das so entstehende Dreieck Δ_S geht aus dem Dreieck Δ_A der Ankreismitten durch eine Halbdrehung um den FEUERBACH-Punkt von Δ hervor. Die Dreiecke Δ_A und Δ_S haben die EULER-Gerade und den Neunpunktekreis von FEUERBACH gemeinsam. In der vorliegenden Arbeit werden diese und weitere damit in Zusammenhang stehende Resultate mit elementaren Mitteln bewiesen.

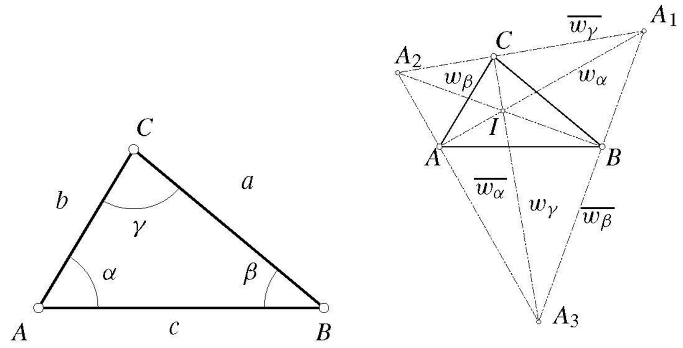

Fig. 1 Notations used in the paper

2 Results and theorems

Now we draw some normals emanating from the excenters A_i and the incenter I. We use the symbol $n_{A_1}(AC)$ to indicate that this line is perpendicular to AC and contains the point A_1. Drawing the lines $n_{A_1}(AC)$, $n_{A_2}(BC)$ and $n_I(AB)$, respectively, we observe that these lines are concurrent in one point S_3. Cyclic rearrangement of (A, B, C) and $(1, 2, 3)$ enables us to state the following theorem:

Theorem 2.1 *The following triples of lines are concurrent:*

(1) $(n_{A_1}(AC), n_{A_2}(BC), n_I(AB))$ *are concurrent in* S_3.
(2) $(n_{A_2}(AB), n_{A_3}(AC), n_I(BC))$ *are concurrent in* S_1.
(3) $(n_{A_3}(BC), n_{A_1}(AB), n_I(AC))$ *are concurrent in* S_2.

Even in classical literature [3, 4] these points and the concurrencies of these normals are not mentioned. The concurrencies of the lines mentioned in Theorem 2.1 are illustrated in Fig. 2.

Moreover, we are able to prove the following result:

Theorem 2.2

(1) *The circumcenter of the triangle* $\Delta_S := \{S_1, S_2, S_3\}$ *is the incenter of* Δ.
(2) *The circumradius of* Δ_S *equals twice the circumradius of* Δ.

For the sake of simplicity we use the abbreviation $\Delta_A := \{A_1, A_2, A_3\}$ and state:

Theorem 2.3 *The triangles* Δ_A *and* Δ_S *are congruent. There exists a rotation* ρ *about the center of the* FEUERBACH *circle of* Δ_S *with angle* $\phi = \pi$ *with* $\rho(\Delta_A) = \Delta_S$.

Theorem 2.4

(1) *The* FEUERBACH *circle of* Δ_S *equals the circumcircle of* Δ.
(2) *The triangles* Δ_A *and* Δ_S *share the* FEUERBACH *circle.*

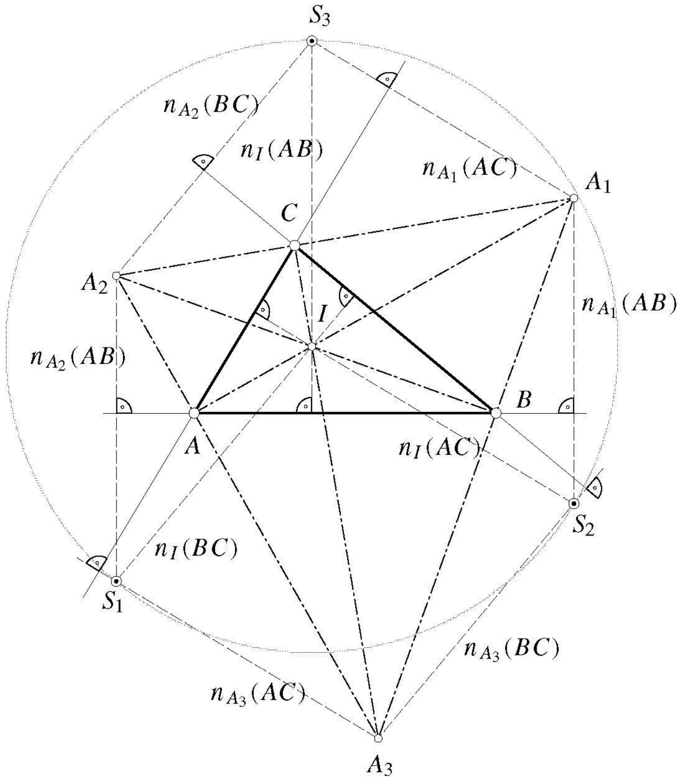

Fig. 2 Three remarkable points occuring as the intersection of some normals

Theorem 2.5

(1) *The incenter I lies on the* EULER *line e_{Δ_S} of Δ_S.*

(2) *The triangles Δ_A and Δ_S share the* EULER *line.*

None of the above theorems are hitherto known. Even in [3, 4] the points S_i and the theorems dealing with them are not mentioned.

3 Proof of the main results

Proof of Theorem 2.1. In order to show that the lines $n_{A_1}(AC)$, $n_{A_2}(BC)$ and $n_I(AB)$ are concurrent in S_3, we compute the length of IS_3 in two different ways and obtain equal results. $\overline{IS_3}$ can be seen as the coordinate of the intersection points $n_I(AB) \wedge n_{A_1}(AC)$ and $n_I(AB) \wedge n_{A_2}(BC)$ on $n_I(AB)$.

We look at the triangles appearing in Fig. 3 and compute the length $\overline{IS_3}$. The first triangle to look at is $\Delta_1 := \{A, B, A_1\}$. The lengths of its edges are $\overline{AB} = c$, $\overline{BA_1}$ and $\overline{A_1A}$,

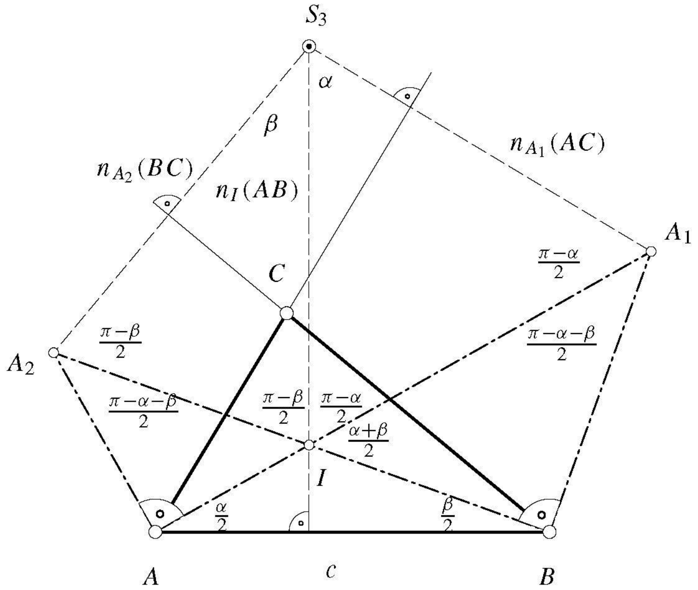

Fig. 3 Computation of $\overline{IS_3}$

respectively. The opposite angles have values $\frac{1}{2}(\pi - \alpha - \beta)$, $\alpha/2$ and $\frac{1}{2}(\pi + \beta)$. So we find

$$\overline{A_1B} = c\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha+\beta}{2}}. \tag{1}$$

The next triangle we pay attention to is $\Delta_2 := \{B, I, A_1\}$. The lengths of edges appearing here are $\overline{BI}$, $\overline{IA_1}$ and $\overline{A_1B}$, respectively. The values of the angles lying opposite to these three edges are $\frac{1}{2}(\pi - \alpha - \beta)$, $\pi/2$ and $\frac{1}{2}(\alpha + \beta)$, respectively. Thus we find

$$\overline{IA_1} = 2c\frac{\sin\frac{\alpha}{2}}{\sin(\alpha+\beta)}. \tag{2}$$

At last we look at $\Delta_3 := \{I, A_1, S_3\}$ with edge lengths $\overline{IA_1}$, $\overline{A_1S_3}$ and $\overline{S_3I}$, respectively. The angles opposite to these edges have values α, $\frac{1}{2}(\pi - \alpha)$ and $\frac{1}{2}(\pi - \alpha)$, respectively. Finally, we arrive at

$$\overline{IS_3} = \frac{c}{\sin(\alpha+\beta)}. \tag{3}$$

The computation of $\overline{IS_3}$ can be done in the same way with the triangles $\Delta_1' := \{A, B, A_2\}$, $\Delta_2' := \{A, I, A_2\}$ and $\Delta_3' := \{I, A_2, S_3\}$, which leads again to (3). Thus the coordinate of $n_I(AB) \wedge n_{A_1}(AC)$ and $n_I(AB) \wedge n_{A_2}(BC)$ on $n_I(AB)$ are equal and we have $n_I(AB) \wedge n_{A_1}(AC) = n_I(AB) \wedge n_{A_2}(BC) = S_3$.

Cyclic rearrangement of indices shows that the points S_1 and S_2 mentioned in Theorem 2.1 do exist and lie on the respective three normals. □

We additionally obtain:

Theorem 3.1 *The point* S_i *is the circumcenter of the triangle* $\{I, A_j, A_k\}$ *and* (i, j, k) *is either* $(1, 2, 3)$ *or* $(2, 3, 1)$ *or* $(3, 1, 2)$.

Proof. Looking at triangles $\Delta_4 := \{I, A_1, S_3\}$ and $\Delta_5 := \{I, A_2, S_3\}$ we find

$$\angle I A_1 S_3 = \angle A_1 I S_3 \quad \text{and} \quad \angle I A_2 S_3 = \angle A_2 I S_3 \,.$$

So we have $\overline{IS_3} = \overline{A_2S_3} = \overline{A_1S_3}$. Rearranging the indices completes the proof. □

Proof of Theorem 2.2. Replacing (A, B, C) and $(1, 2, 3)$ cyclically in (3) we obtain

$$\overline{IS_1} = \frac{a}{\sin(\beta+\gamma)} \quad \text{and} \quad \overline{IS_2} = \frac{b}{\sin(\alpha+\gamma)} \,. \tag{4}$$

Since the values of the interior angles sum up to π, that is $\gamma = \pi - \alpha - \beta$, we find

$$\frac{c}{\sin(\alpha+\beta)} = \frac{c}{\sin\gamma} = 2R \,. \tag{5}$$

Further, we use the well-known formulae

$$\frac{c}{\sin\gamma} = \frac{b}{\sin\beta} = \frac{a}{\sin\alpha} = 2R \,, \tag{6}$$

which gives a simple relation between the angles, the side lengths and the circumradius R of Δ. Thus the circumradius of $\{S_1, S_2, S_3\}$ is twice the circumradius of Δ. □

Proof of Theorem 2.3. In order to show that $\Delta_A \equiv \Delta_S$, we show that the lines A_1A_2 and S_1S_2 are parallel. (Equivalently, we could show that A_1A_3 and S_1S_3 are parallel and also A_3A_2 and S_3S_2 are parallel. Changing the indices while keeping the cycling ordering we obtain the equivalent results for the other pairs of lines.)

By definition we have $A_1A_2 \perp w_\gamma$ and from Theorem 2.2 we have $\overline{IS_1} = \overline{IS_2}$. Since w_γ is interior bisector of AC and BC it also is interior bisector of $n_I(AC)$ and $n_I(BC)$. Consequently, $S_1S_2 \perp w_\gamma$ and thus S_1S_2 is parallel to A_1A_2.

Since $n_{A_2}(AB)$ and $n_{A_1}(AB)$ are parallel we have $\overline{A_1A_2} = \overline{S_1S_2}$. The same is true if we change indices $(1, 2, 3)$ and (A, B, C), respectively, while keeping the cyclic ordering.

So far we have shown that Δ_A is congruent to Δ_S. Now we have to prove that there is a rotation ρ with angle π and $\rho(\Delta_A) = \Delta_S$.

We observe that A_2S_2 and A_3S_3 are the diagonals of the parallelogram $\Pi_1 := \{S_2, S_3, A_2, A_3\}$. Thus they intersect in a point X. Each of the parallelograms $\Pi_2 := \{S_1, S_3, A_1, A_3\}$ and $\Pi_3 := \{S_1, S_2, A_1, A_2\}$ shares a diagonal with Π_1. Therefore, the diagonals of Π_1,

Π_2 and Π_3, respectively, are concurrent in X. Consequently, there exists a unique reflection about X which maps Δ_A to Δ_S. The existence of this reflection is equivalent to the existence of a rotation ρ about X with angle π transforming Δ_A into Δ_S.

At last we have to show that X is the FEUERBACH point F_{Δ_S} of Δ_S. The base triangle Δ is the pedal triangle of Δ_A. Thus the circumcircle of Δ is the FEUERBACH circle of Δ_A. Since ρ maps Δ_A to Δ_S it maps the corresponding pedal triangles onto each other by reflecting them about X. Thus the FEUERBACH circles of Δ_S and Δ_A coincide such as their centers coincide in X. □

Proof of Theorem 2.4. There is nothing to be done. This theorem is a consequence of the proof of Theorem 2.3. □

Proof of Theorem 2.5. The incenter I of Δ is the circumcenter of Δ_S, see Theorem 2.1. Thus it is contained in the EULER line e_{Δ_S} of Δ_S.

Since e_{Δ_S} passes through the FEUERBACH point F_{Δ_S}, the rotation ρ with center F_{Δ_S} transforms e_{Δ_S} into e_{Δ_A}. □

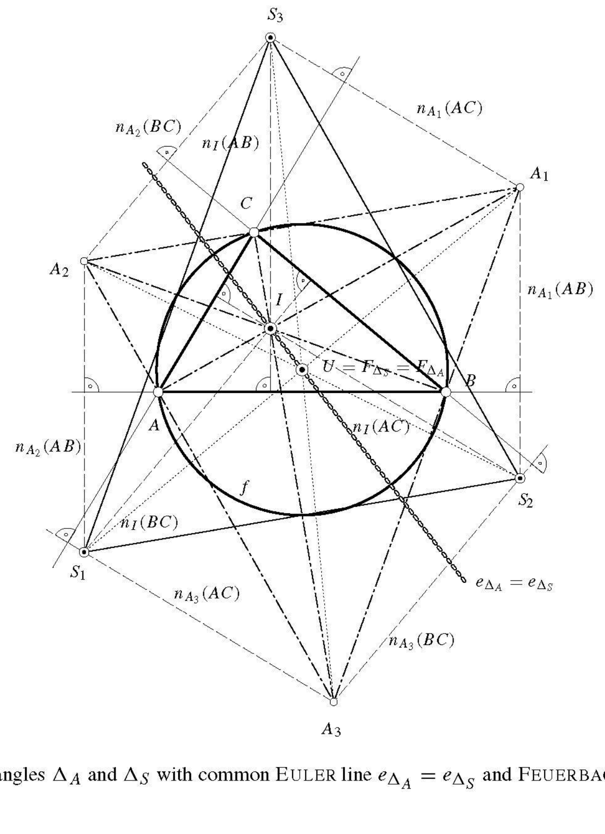

Fig. 4 Triangles Δ_A and Δ_S with common EULER line $e_{\Delta_A} = e_{\Delta_S}$ and FEUERBACH circle f

4 Alternative interpretation

As remarked in Section 1, the theorems given in Section 2 can be seen in a different light.

For a given triangle $\Delta_A := \{A_1, A_2, A_3\}$ draw the orthoptic triangle $\Delta_O := \{B_1, B_2, B_3\}$, where $B_i \in A_j A_k$ with cyclic ordering of (i, j, k). The orthocenter H of Δ_A is the incenter of Δ_O. Now we recall that Δ_A is the excenter triangle of Δ_O.

Thus Theorem 2.1 can be reformulated:

Theorem 4.1 (*Equivalent to Theorem* 2.1)

The normals from the vertex A_1 of the base triangle Δ_A to the side B_1B_2 of the orthoptic triangle Δ_O, the normal from A_3 to B_2B_3, and the normal through the orthocenter H of Δ_A to B_1B_3 are concurrent in a point S_2.

This remains true if we change the indices while keeping the cyclic ordering. □

Acknowledgements. I have to express my sincere thanks to the reviewer for his hints and comments.

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