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1 Introduction

A (topological) disk is a subset of the euclidean plane homeomorphic to the unit ball. If two disks have a common interior point then we say that the disks overlap. A sequence $\mathcal{C} = \langle C_1, \ldots, C_n \rangle$ of mutually non overlapping congruent disks where $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$ is called a snake. If the snake $\mathcal{C}$ is not a proper subset of another snake of disks congruent to the members of $\mathcal{C}$ then we say that the snake is limited.

We are concerned with the following question: What is the minimum number of mutually non overlapping congruent disks which can form a limited snake? Here we prove

Theorem. *The minimum number of mutually non overlapping congruent disks which can form a limited snake is four.*

Surprisingly, under the assumption of convexity the above problem seems to be much more complicated. Fig. 1 shows that six mutually non overlapping congruent copies of a

Auf einem Tisch legt man mit lauter gleichen Münzen eine „Münzschlange“: an eine erste Münze anstossend legt man eine zweite, daran anstossend eine dritte usw. Bei diesem Legespiel kann eine Konfiguration entstehen, bei der man weder am Kopf noch am Schwanz der Schlange eine weitere Münze anschliessen kann, weil der Platz durch andere Münzen des Schlangenkörpers versperrt wird. Welches ist die kleinste Anzahl Münzen, bei der dies vorkommen kann? Die Autoren untersuchen und beantworten die entsprechende Frage, wenn man die runden Münzen durch eine beliebige einfach zusammenhängende beschränkte Menge der Euklidischen Ebene ersetzt. Das entsprechende Problem für konvexe beschränkte Mengen ist hingegen noch offen.

certain rectangle can form a limited snake. Do there exist convex disks whose less then six mutually non overlapping congruent copies could form limited snakes? We conjecture that the answer to this question is in the negative.

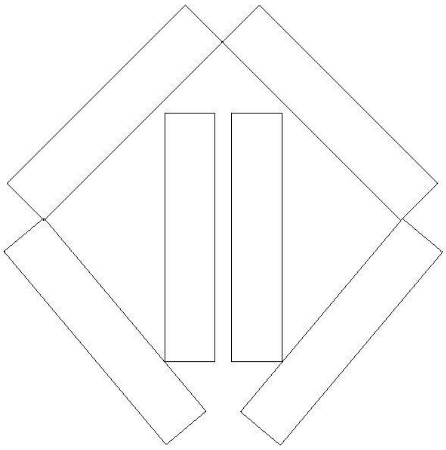

Fig. 1

Also, the problem of determining the minimum number of mutually non overlapping congruent copies of a given disk which can form a limited snake is very complicated. The only known result in this direction is that the minimum number of mutually non overlapping congruent balls which can form a limited snake is ten (see [1]).

For additional results on more restrictive variants of the snake problem, see [2, 3, 4, 5, 6, 7, 8].

2 Proof of the theorem

Fig. 2 shows that this minimum number is at most four.

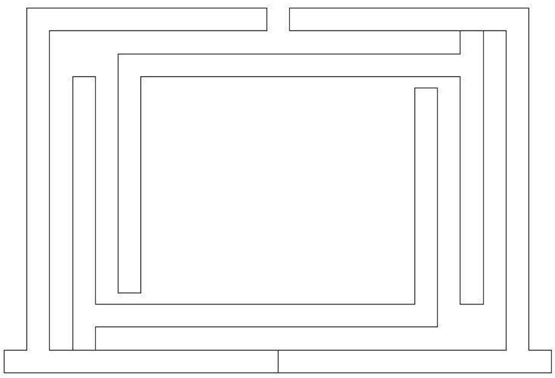

Fig. 2

To complete the proof we have to show that two or three mutually non overlapping congruent disks cannot form a limited snake. Let C be an arbitrary disk and let $\mathcal{C} = \langle C_1, \ldots, C_n \rangle$ be a limited snake consisting of disks congruent with C.

We start with the case $n = 2$. First assume that $\operatorname{conv} C_1 = \operatorname{conv} C_2$, i.e., the convex hulls of C_1 and C_2 coincide. If every boundary point of $\operatorname{conv} C_1$ belongs to C_1, i.e., C_1 is convex, then C_1 and C_2 coincide, which is impossible. Thus there exists a boundary point of $\operatorname{conv} C_1$ which does not belong to C_1. This point lies in the relative interior of a segment joining two extreme points, say A and B, of $\operatorname{conv} C_1$. Recall that a point of a disk is an extreme point of the disk if there exists no segment in the disk that contains the point in its relative interior. The points A and B are extreme points of C_1 and they can be joined with a path P_1 whose points different from A and B lie in the interior of C_1. Also, the points A and B are extreme points of C_2 and they can be joined with a path P_2 whose points different from A and B lie in the interior of C_2. Then either the bounded region surrounded by P_1 and the segment $\overline{AB}$ contains P_2 or the bounded region surrounded by P_2 and the segment $\overline{AB}$ contains P_1, which is impossible since $\operatorname{conv} C_1 = \operatorname{conv} C_2$.

Thus there exists a point P of C_1 which does not belong to $\operatorname{conv} C_2$. Then P can be strictly separated from C_2 by a line l. Let l' be the support line of C_1 which is parallel to l and does not separate C_1 and C_2. Reflecting C_1 with respect to l' we obtain a third copy of C which forms with C_1 and C_2 a three element snake, a contradiction.

Now we turn to the case $n = 3$. Let $\overline{DE}$ be a diameter of C_1 and consider the stripe S_1 whose boundary lines, say l_1 and l_2, go through D and E, respectively, and are perpendicular to $\overline{DE}$. If C_3 is not contained in S_1 then consider the support line l of C_3 which is parallel to l_1 and whose distance from S_1 is maximal. Without loss of generality we may assume that l_2 separates l and l_1. Let F be a common point of C_3 and l. The disk C_2 cannot intersect both l and l_1 since the distance between the two lines is greater than the diameter of C_2. Thus either reflecting C_1 with respect to l_1 or reflecting C_3 with respect to l we obtain a fourth copy of C which forms with C_1, C_2 and C_3 a four element snake, a contradiction.

Thus C_3 lies in S_1. Let $\overline{GH}$ be a diameter of C_3 and consider the stripe S_3 whose boundary lines, say l_3 and l_4, go through G and H, respectively, and are perpendicular to $\overline{GH}$. If C_1 is not contained in S_3 then repeating the previous argument we obtain a contradiction. Therefore C_1 lies in S_3. If $S_1 = S_3$, i.e., $l_1 = l_3$ and $l_2 = l_4$ without loss of generality, then D and G are different points since $C_1 \cap C_3 = \emptyset$. Now C_2 does not contain both D and H since their distance is greater than the diameter of C_2. Therefore either reflecting C_1 with respect to l_1 or reflecting C_3 with respect to l_2 we obtain a fourth copy of C which forms with C_1, C_2 and C_3 a four element snake, a contradiction. On the other hand, if S_1 and S_2 are different stripes then their intersection is a parallelogram which contains both C_1 and C_3. The points D and E can be joined by a path P_3 in C_1 while G and H can be joined by a path P_4 in C_3. Since the pathes join opposite sides of the above parallelogram they necessarily intersect each other. But this is impossible since C_1 and C_3 are disjoint. This completes the proof of the theorem.

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