Elem. Math. 64 (2009) 45 – 52
0013-6018/09/020045-8

**Elemente der Mathematik**

# Formulas giving prime numbers under Cramér's conjecture

Bakir Farhi

Bakir Farhi received his M.Sc. and his Ph.D. in mathematics from the Université Paris 6 in 1999 and 2003, respectively. He was an assistant professor at the Université de Bretagne Occidentale in 2003/04. Since 2004 he is an assistant professor at the Université du Maine, Le Mans.

## 1 Introduction

Throughout this article, we let $[x]$ denote the integer part of a given real number $x$; also, we let $(p_n)_{n\in\mathbb{N}}$ denote the sequence of all prime numbers and we set $\Delta p_n := p_{n+1} - p_n$ for any $n \in \mathbb{N}$. Further, if $A$ is a subset of $\mathbb{R}$ and $x$ is a real number, we will let $A + x$ denote the subset of $\mathbb{R}$ defined by $A + x := \{a + x \mid a \in A\}$.

In [4], Mills proved the existence of an absolute constant $A > 1$ for which $[A^{3^n}]$ is a prime number for any positive integer $n$ and in [6], Wright proved the existence of an absolute constant $\alpha > 0$ for which the infinite sequence $[\alpha]$, $[2^\alpha]$, $[2^{2^\alpha}]$, ... is composed of prime numbers. Let us describe the method used by these two authors. They start from an upper bound for $\Delta p_n$ as a function of $p_n$. Such an upper bound allows to construct an increasing function $h$ (more or less elementary, according to the used upper bound of $\Delta p_n$) such that

Ein klassisches Problem der Zahlentheorie ist die Suche nach einfachen Formeln zur Erzeugung von Primzahlen. So bewies W.H. Mills im Jahr 1947, dass eine Konstante $A > 1$ existiert, so dass die natürliche Zahl $[A^{3^n}]$ für alle positiven natürlichen Zahlen $n$ eine Primzahl ist; hierbei bedeutet $[x]$ den ganzzahligen Anteil der reellen Zahl $x$. Vier Jahre später wies der Zahlentheoretiker E.M. Wright die Existenz einer Konstanten $\alpha > 0$ nach, so dass die Folge $[\alpha]$, $[2^\alpha]$, $[2^{2^\alpha}]$, ... aus lauter Primzahlen besteht. In dem nachfolgenden Beitrag gelingt es dem Autor, unter der Annahme der Cramérschen Vermutung zu vorgegebenem $\xi > 1$ jeweils eine reelle Zahl $A = A(\xi) > 1$ zu konstruieren, so dass die Grösse $[A^{n^\xi}]$ für alle $n \in \mathbb{N}$, $n > 0$, eine Primzahl ist. Das Interessante an dieser Konstruktion ist, dass die auf diese Weise erzeugte Primzahlfolge deutlich langsamer als die von Mills und Wright gegebenen Folgen wächst.

between any two consecutive terms of the sequence $(h(n))_n$, there is at least one prime number. Setting $f_n := h \circ \ldots \circ h$ (where $h$ is applied $n$ times), they deduce from the last fact, the existence of a real constant $A$ for which the sequence $([f_n(A)])_n$ consists of prime numbers.

With this method, Wright used the upper bound $\Delta p_n \leq p_n$, which is nothing else than Bertrand's postulate, and Mills used Ingham's upper bound $\Delta p_n \leq p_n^{5/8+\varepsilon}$, which is valid for any $n$ sufficiently large depending on the given $\varepsilon > 0$. The functions $h$, which are derived from these upper bounds, are $h(x) = 2^x$ for Wright and $h(x) = x^3$ for Mills. Then, the theorems of [4] and [6] follow.

Notice that the more the upper bound of $\Delta p_n$ is refined, the more the function $h$ will be smaller and the more the obtained sequence of prime numbers will grow slowly (for instance, the sequence of Mills grows more slowly than Wright's one). From this fact, in order to have a sequence of prime numbers which grows even more slowly, we must use more refined upper bounds for $\Delta p_n$. But up to now even the powerful Riemann hypothesis gives only the estimate $\Delta p_n = O(p_n^{1/2} \log p_n)$. A famous conjecture (which is a little too strong compared with the last estimate) states that between two consecutive squares, there is always a prime number (see [2]). So, according to this conjecture, the function $h(x) = x^2$ is admissible for the method described above, which permits to conclude the existence of a constant $B > 1$ for which $[B^{2^n}]$ is a prime number for any positive integer $n$. We thus obtain (assuming this conjecture), a sequence of prime numbers growing more slowly than Mills' one.

Based on heuristic and probabilistic arguments, Cramér [1] was led to the conjecture that $\Delta p_n = O(\log^2 p_n)$; note that it is known that $\Delta p_n = O(\log p_n)$ cannot hold (see [5]). Thus, by taking for the method described above $h(x) = c \log^2 x$ $(c > 0)$, we obtain (via Cramér's conjecture) sequences of prime numbers having an explicit form and growing much more slowly than Mills' one. The inconvenience of this application is that the explicit form in question $[f_n(A)]$ is not elementary, because $f_n$ does not have a simple expression as a function of $n$.

To overcome this problem, we were led to generalize Mills' method by considering instead of one function $h$, a sequence of functions $(h_m)_m$ and, hence, in this situation $f_n$ is rather the composition of $n$ functions $h_0, \ldots, h_{n-1}$. This allows to give for $f_n$ the form which we want, and if we set $h_n := f_{n+1} \circ f_n^{-1}$, we have only to check whether it is true that for any $n$ and any $x$ sufficiently large (relative to $n$), the interval $[h_n(x), h_n(x+1) - 1[$ contains at least one prime number or not. In the affirmative case, we will deduce the existence of a real number $A$ for which the formula $[f_n(A)]$ gives a prime number for any positive integer $n$ (see Theorem 1 and its proof).

Under a conjecture weaker than Cramér's one, we derive from this generalization two new types of explicit formulae giving prime numbers. We also give other applications of our main result (outside the subject of prime numbers) and we conclude this article by some open questions related to the results which we obtain.

## 2 Results

The main result of this article is the following theorem.

**Theorem 1** *Let $I = ]a, b[$ (with $a, b \in \overline{\mathbb{R}}$, $a < b$) be an open interval of $\mathbb{R}$, $n_0$ a non-negative integer and $(f_n)_{n \geq n_0}$ a sequence of real functions, which are differentiable and increasing on $I$.*

*Assume that the functions $f'_{n+1}/f'_n$ ($n \geq n_0$) are non-decreasing on $I$ and that for all $x \in I$, the sequence $(f_n(x))_{n \geq n_0}$ is increasing. Further assume that there exists a real function $g$, non-decreasing on $\mathbb{R}$ and verifying*

$$(g \circ f_{n+1})(x) \leq \frac{f'_{n+1}}{f'_n}(x) \qquad (\forall n \geq n_0, \ \forall x \in I). \tag{1}$$

*Then, for any sequence of integers $(u_n)_n$, verifying $\limsup_{n \to +\infty} u_n = +\infty$,*

$$u_{n+1} - u_n \leq g(u_n) - 1 \qquad (\forall n \geq n_1), \tag{2}$$

*and for which at least one of the terms $u_n$ belongs to $f_{n_0}(I) \cap (f_{n_0}(I) - 1)$, there exists a real $A \in I$, for which the sequence $([f_n(A)])_{n \geq n_0}$ is an increasing subsequence of $(u_n)_n$.*

*Proof*. By shifting, if necessary, the sequence of functions $(f_n)_{n \geq n_0}$, we may assume that $n_0 = 0$ and by shifting, if necessary, the sequence $(u_n)_n$, we may assume that we have

$$u_{n+1} - u_n \leq g(u_n) - 1 \qquad (\forall n \in \mathbb{N}). \tag{2'}$$

We begin the proof by some remarks and preliminary notations which allow to simplify the situation of the theorem.

Since the function $f_n$ for given $n \in \mathbb{N}$ is assumed to be differentiable (hence continuous) and increasing on $I = ]a, b[$, it is a bijection from $I$ onto $f_n(I) = ]\lambda_n, \mu_n[$, where $\lambda_n := \lim_{x \to a} f_n(x)$ and $\mu_n := \lim_{x \to b} f_n(x)$ ($\lambda_n$ and $\mu_n$ belong to $\overline{\mathbb{R}}$). Now, let us introduce the following functions

$$h_n : ]\lambda_n, \mu_n[ \longrightarrow ]\lambda_{n+1}, \mu_{n+1}[ \ \text{ defined by } \ h_n := f_{n+1} \circ f_n^{-1} \qquad (\forall n \in \mathbb{N}).$$

Since the functions $f_n$ and $f_{n+1}$ for given $n \in \mathbb{N}$, are differentiable and increasing on $I$, the function $h_n$ is differentiable and increasing on $]\lambda_n, \mu_n[$. Further, the hypothesis of the theorem concerning the growth of the sequence $(f_n(x))_n$ ($x \in I$) amounts to

$$h_n(x) > x \qquad (\forall n \in \mathbb{N}, \ \forall x \in ]\lambda_n, \mu_n[). \tag{3}$$

Next, let us show that for any $n \in \mathbb{N}$, the function $h_n$ is convex on $]\lambda_n, \mu_n[$. To do this, we check that the derivative $h'_n$ ($n \in \mathbb{N}$) is non-decreasing on the interval $]\lambda_n, \mu_n[$. Given $n \in \mathbb{N}$, we have

$$h'_n = (f_{n+1} \circ f_n^{-1})' = (f_n^{-1})' \cdot f'_{n+1} \circ f_n^{-1} = \frac{f'_{n+1} \circ f_n^{-1}}{f'_n \circ f_n^{-1}} = \frac{f'_{n+1}}{f'_n} \circ f_n^{-1}.$$

Since the function $f'_{n+1}/f'_n$ is non-decreasing on $I$ and the function $f_n^{-1}$ is increasing on $f_n(I) = ]\lambda_n, \mu_n[$ the function $h'_n$ (as a composite of two non-decreasing functions), is non-decreasing on $]\lambda_n, \mu_n[$. So the function $h_n$ is effectively convex on $]\lambda_n, \mu_n[$.

The rest of the proof consists of the following three steps:

*1st Step*: We are going to show that we have

$$(g \circ h_n)(y) \leq h_n(y+1) - h_n(y) \qquad (\forall n \in \mathbb{N},\ \forall y \in ]\lambda_n, \mu_n - 1[). \tag{4}$$

In fact, we will see later that the interval $]\lambda_n, \mu_n - 1[$ is never empty. Let $n \in \mathbb{N}$ and $y \in ]\lambda_n, \mu_n - 1[$ be fixed and set $x := f_n^{-1}(y)$. The convexity of $h_n$ on $]\lambda_n, \mu_n[$, proved above, implies that we have

$$h_n(u) \geq h_n'(t)(u-t) + h_n(t) \qquad (\forall t, u \in ]\lambda_n, \mu_n[).$$

By taking in this last inequality $t = y$ and $u = y + 1$, we obtain

$$\begin{aligned} h_n(y+1) - h_n(y) &\geq h_n'(y) \\ &= \left(\frac{f_{n+1}'}{f_n'}\right)(x) && \left(\text{because } h_n' = \frac{f_{n+1}'}{f_n'} \circ f_n^{-1} \text{ and } x = f_n^{-1}(y)\right) \\ &\geq (g \circ f_{n+1})(x) && (\text{from hypothesis (1) of the theorem}) \\ &= (g \circ f_{n+1} \circ f_n^{-1})(y) \\ &= (g \circ h_n)(y). \end{aligned}$$

The relation (4) now follows.

*2nd Step*: We are going to construct an increasing sequence $(k_n)_{n \in \mathbb{N}}$ of non-negative integers such that the subsequence of $(u_n)_n$ with general term $v_n = u_{k_n}$ satisfies

$$\begin{cases} v_n \in ]\lambda_n, \mu_n - 1[\,, \\ h_n(v_n) \leq v_{n+1} < h_n(v_n + 1) - 1 \end{cases} \qquad (\forall n \in \mathbb{N}).$$

We proceed by induction as follows:

- We pick $k_0 \in \mathbb{N}$ such that $u_{k_0} \in f_0(I) \cap (f_0(I) - 1) = ]\lambda_0, \mu_0 - 1[$. Notice that the existence of such an integer $k_0$ is a hypothesis of the theorem.
- If, for some $n \in \mathbb{N}$, an integer $k_n \in \mathbb{N}$ is chosen such that $u_{k_n} \in ]\lambda_n, \mu_n - 1[$, let

$$X_n := \left\{k \in \mathbb{N} \mid k > k_n \text{ and } u_k \geq h_n(u_{k_n})\right\}.$$

From the hypothesis $\limsup_{n \to +\infty} u_n = +\infty$, the subset $X_n$ of $\mathbb{N}$ is non-empty, it thus admits a smallest element which we call $k_{n+1}$. So, we have

$$k_{n+1} > k_n, \quad u_{k_{n+1}} \geq h_n(u_{k_n}), \quad \text{and} \quad k_{n+1} - 1 \notin X_n.$$

We claim that the facts "$k_{n+1} > k_n$" and "$k_{n+1} - 1 \notin X_n$" imply

$$u_{k_{n+1}-1} < h_n(u_{k_n}). \tag{5}$$

Indeed, either $k_{n+1} = k_n + 1$, in which case we have $u_{k_{n+1}-1} = u_{k_n} < h_n(u_{k_n})$ from (3), or $k_{n+1} > k_n + 1$, that is $k_{n+1} - 1 > k_n$. But since $k_{n+1} - 1 \notin X_n$, we must have $u_{k_{n+1}-1} < h_n(u_{k_n})$, as required. It follows

$$\begin{aligned} u_{k_{n+1}} &\leq u_{k_{n+1}-1} + g(u_{k_{n+1}-1}) - 1 && \text{(from (2'))} \\ &< h_n(u_{k_n}) + (g \circ h_n)(u_{k_n}) - 1 && \text{(using (5) and since } g \text{ is non-decreasing)} \\ &\leq h_n(u_{k_n} + 1) - 1 && \text{(from (4)).} \end{aligned}$$

Hence, we have

$$u_{k_{n+1}} < h_n(u_{k_n} + 1) - 1,$$

and thus

$$h_n(u_{k_n}) \leq u_{k_{n+1}} < h_n(u_{k_n} + 1) - 1.$$

Since the function $h_n$ takes its values in $]\lambda_{n+1}, \mu_{n+1}[$, the last inequality shows that $u_{k_{n+1}} \in ]\lambda_{n+1}, \mu_{n+1} - 1[$. This ensures that the induction process works and gives the required sequence $(k_n)_n$. Notice also that the subsequence $(v_n)_n$ of $(u_n)_n$, which we have just constructed, is increasing because we have $v_{n+1} \geq h_n(v_n) > v_n$ by (3) for any $n \in \mathbb{N}$.

*3rd Step*: To conclude the proof, we will show the existence of a real $A \in I$, for which we have $v_n = [f_n(A)]$ for any $n \in \mathbb{N}$. To do this, we introduce two real sequences $(x_n)_n$ and $(y_n)_n$, with elements in $I$, which we define by

$$x_n := f_n^{-1}(v_n) \quad \text{and} \quad y_n := f_n^{-1}(v_n + 1) \qquad (\forall n \in \mathbb{N}).$$

Since the functions $f_n$ are increasing, we have $x_n < y_n$ for all $n \in \mathbb{N}$. We claim that the sequence $(x_n)_n$ is non-decreasing and that the sequence $(y_n)_n$ is decreasing. Indeed, for any $n \in \mathbb{N}$, we have

$$x_n = f_n^{-1}(v_n) = (f_{n+1}^{-1} \circ h_n)(v_n) \leq f_{n+1}^{-1}(v_{n+1}) = x_{n+1}$$

and

$$y_n = f_n^{-1}(v_n + 1) = (f_{n+1}^{-1} \circ h_n)(v_n + 1) > f_{n+1}^{-1}(v_{n+1} + 1) = y_{n+1}.$$

In these last relations, we have just used the facts that $f_{n+1}^{-1}$ is increasing and $h_n(v_n) \leq v_{n+1} < h_n(v_n + 1) - 1$. The intervals $[x_n, y_n]$ $(n \in \mathbb{N})$ are thus nested intervals of $\mathbb{R}$. Consequently, their intersection is non-empty according to Cantor's intersection theorem. Pick $A$ an arbitrary real number belonging to this intersection, i.e., $x_n \leq A \leq y_n$ for all $n \in \mathbb{N}$, in particular $A \in I$. In fact, $A$ verifies even

$$x_n \leq A < y_n \qquad (\forall n \in \mathbb{N}),$$

because if $A = y_m$ for some $m \in \mathbb{N}$, we will have, since the sequence $(y_n)_n$ decreases, $A > y_{m+1}$, contradicting the inequality $A \leq y_{m+1}$. It follows from the growth of the functions $f_n$ that we have

$$f_n(x_n) \leq f_n(A) < f_n(y_n) \qquad (\forall n \in \mathbb{N}),$$

that is

$$v_n \leq f_n(A) < v_n + 1 \qquad (\forall n \in \mathbb{N}).$$

Then, since $v_n$ is an integer for all $n \in \mathbb{N}$, we conclude

$$[f_n(A)] = v_n \qquad (\forall n \in \mathbb{N}).$$

This completes the proof. □

**Remarks.** Mills' theorem [4] can be recovered by applying Theorem 1 for $I = ]1, +\infty[$, $n_0 = 0$, $f_n(x) = x^{3^n}$ $(n \in \mathbb{N}, x \in I)$, $g(x) = x^{2/3}$, if $x > 0$ and $g(x) = 0$, if $x \leq 0$, and $(u_n)_n$ the sequence of prime numbers. In this application, we check relation (1) of Theorem 1 by simple calculus and we deduce relation (2) from Ingham's estimate quoted in the introduction. The remaining hypotheses of Theorem 1 are immediately verified.

Wright's theorem [6] can also be recovered, by applying Theorem 1 for $I = ]0, +\infty[$, $n_0 = 0$, $(f_n)_n$ the sequence of functions which is defined on $I$ by $f_0 = \mathrm{Id}_I$ and $f_{n+1} = 2^{f_n}$ $(n \in \mathbb{N})$, $g(x) = (\log 2)x$ $(\forall x \in \mathbb{R})$, and $(u_n)_n$ the sequence of prime numbers. In order to check relation (1) of Theorem 1, note that we have $f'_{n+1}/f'_n = (\log 2) f_{n+1}$ for any $n \in \mathbb{N}$. Relation (2) is a consequence of the prime number theorem, but it can be obtained by using elementary arguments due to Chebyshev (see [3]). The remaining hypothesis of Theorem 1 is immediately verified.

**N.B.** In the above two applications of Theorem 1, the sequence of functions $(h_n)_n$ introduced in the proof is constant. Indeed, for the first application, we have $h_n(x) = x^3$ $(n \in \mathbb{N})$ and for the second one, we find $h_n(x) = 2^x$ $(n \in \mathbb{N})$. As explained in the introduction, the possibility of taking $(h_n)_n$ not constant is the crucial point of our approach. In the following, we are going to give some applications of Theorem 1 in which the sequence $(h_n)_n$ is not constant. If we admit the following conjecture (which is weaker than Cramér's one [1]), we obtain two new types of explicit sequences of prime numbers, which grow much more slowly than the ones of Mills and Wright.

**Conjecture 2** *There exists an absolute constant $k > 1$ such that*

$$\Delta p_n = O\left((\log p_n)^k\right).$$

Under this conjecture, we obtain by applying Theorem 1, the following two corollaries.

**Corollary 3** *Assuming Conjecture 2, there exists for all real numbers $\xi > 1$, a real number $A = A(\xi) > 1$, for which the sequence $([A^{n^\xi}])_{n \geq 1}$ is an increasing sequence of prime numbers.*

*Proof*. Let $\xi > 1$ be fixed, $k > 1$ an admissible constant as in Conjecture 2, and $a > 1$ a real number such that

$$(\log x)^{k+1} \leq x^{1/2} \qquad (\forall x > a), \tag{6}$$

$$(n+1)^{k+1} \leq a^{n^{\xi-1}/2} \qquad (\forall n \geq 1). \tag{7}$$

Such an $a$ exists because

$$\lim_{x \to +\infty} (\log x)^{k+1}/x^{1/2} = 0 \quad \text{and} \quad \lim_{n \to +\infty} (n+1)^{2(k+1)/n^{\xi-1}} = 1.$$

We apply Theorem 1 for $I = ]a, +\infty[$, $n_0 = 1$, $f_n(x) = x^{n^\xi}$ $(n \geq 1, x \in I)$, $g(x) = (\log x)^{k+1}$, if $x > 1$, and $g(x) = 0$, if $x \leq 1$, and $(u_n)_n$ the sequence of prime numbers. Let us check the hypotheses of Theorem 1.

The functions $f_n$ are clearly increasing and differentiable on $I$. We have $f_n'(x) = n^\xi x^{n^\xi - 1}$, therefore

$$\frac{f_{n+1}'}{f_n'}(x) = \left(\frac{n+1}{n}\right)^\xi x^{(n+1)^\xi - n^\xi} \qquad (\forall n \geq 1,\ \forall x \in I).$$

We thus see that the functions $f_{n+1}'/f_n'$ $(n \geq 1)$ are non-decreasing on $I$. Further, if $x$ is a fixed real in $I$, the sequence $(f_n(x))_{n \geq 1}$ is clearly increasing. Now, we have for any integer $n \geq 1$ and for any real $x \in I$:

$$\begin{aligned} g \circ f_{n+1}(x) &= (n+1)^{\xi(k+1)}(\log x)^{k+1} \\ &\leq a^{\xi n^{\xi-1}/2} x^{1/2} && \text{(from (6) and (7))} \\ &\leq x^{\xi n^{\xi-1}} && (\text{because } x > a \text{ and } \xi n^{\xi-1} > 1) \\ &\leq x^{(n+1)^\xi - n^\xi} && (\text{because } \xi n^{\xi-1} \leq (n+1)^\xi - n^\xi) \\ &\leq \frac{f_{n+1}'}{f_n'}(x). \end{aligned}$$

Relation (1) of Theorem 1 now follows. Next, relation (2) of Theorem 1 follows immediately from Conjecture 2. Finally, $f_{n_0}(I) \cap (f_{n_0}(I) - 1) = ]a, +\infty[$ contains prime numbers as large as we want. The hypothesis of Theorem 1 are thus all satisfied, so we can apply this latter to the present situation. Corollary 3 follows from this application. □

**Corollary 4** *Assume that Conjecture 2 is true and let $k > 1$ be an admissible constant in this conjecture. Then, for any positive real number $\varepsilon$, there exists an integer $n_0 = n_0(\varepsilon, k) \geq 1$ and a real number $B = B(\varepsilon, k) > 0$ such that the sequence $([B \cdot n!^{k+\varepsilon}])_{n \geq n_0}$ is an increasing sequence of prime numbers.*

*Proof*. Let $\varepsilon$ be a fixed positive real number. From Conjecture 2 (applied with the constant $k > 1$), there exists a positive real number $c_k$ for which we have

$$p_{n+1} - p_n \leq c_k (\log p_n)^k \qquad (\forall n \in \mathbb{N}). \tag{8}$$

We apply Theorem 1 for $I = ]1, 2[$, $n_0 \geq 2$ an integer (depending on $k$ and $\varepsilon$) which we pick large enough such that

$$c_k\left((k+\varepsilon)(n+1)\log(n+1) + \log 2\right)^k + 1 \leq (n+1)^{k+\varepsilon} \qquad (\forall n \geq n_0), \tag{9}$$

and $f_n(x) = n!^{k+\varepsilon} x$ $(n \geq n_0, x \in I)$, $g(x) = c_k(\log x)^k + 1$, if $x > 1$, and $g(x) = 1$, if $x \leq 1$, and $(u_n)_n$ the sequence of prime numbers. In this situation, we can easily check that the hypotheses of Theorem 1 are all satisfied. We just note that relation (1) follows from (9), relation (2) follows from (8), and the last hypothesis of Theorem 1 concerning the sequence $(u_n)_n = (p_n)_n$ is a consequence of Bertrand's postulate. Corollary 4 follows from this application. □

Apart from the context of the prime numbers, we have the following

**Corollary 5** *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of integers such that*

$$1 \leq \limsup_{n\to+\infty}(u_{n+1} - u_n) < +\infty.$$

*Then, we have:*

(1) *For any positive real number $\lambda$, there exists a real number $A > 1$, for which the sequence $([\lambda A^n])_{n\geq 1}$ is an increasing subsequence of $(u_n)_n$.*

(2) *For any real number $A > \limsup_{n\to+\infty}(u_{n+1} - u_n) + 1$, there exists a positive real number $\lambda$, for which the sequence $([\lambda A^n])_{n\geq 1}$ is an increasing subsequence of $(u_n)_n$.*

**Some open problems related to the preceding study:**

We ask (with or without Cramér's conjecture) the following questions:

(1) Does there exist a real number $A > 1$ for which $[A^n]$ is a prime number for every positive integer $n$? (This corresponds to the case $\xi = 1$ which is excluded from Corollary 3.)

(2) More generally than (1), does there exist a couple of real numbers $(\lambda, A)$, with $\lambda > 0$, $A > 1$, for which $[\lambda A^n]$ is a prime number for every positive integer $n$? (This is related to Corollary 5.)

(3) Does there exist a real number $B > 1$, for which $[B \cdot n!^2]$ is a prime number for every sufficiently large non-negative integer $n$? (This is related to Corollary 4.)

## References

[1] Cramér, H.: On the order of magnitude of the difference between consecutive prime numbers. *Acta Arith.* 2 (1936), 23–46.

[2] Guy, R.K.: *Unsolved problems in Number Theory.* 2nd ed., Springer-Verlag, New York 1994.

[3] Hardy, G.H.; Wright, E.M.: *The theory of numbers.* Oxford Univ. Press, London 1979.

[4] Mills, W.H.: A prime-representing function. *Bull. Amer. Math. Soc.* 53 (1947), 604.

[5] Westzynthius, E.: Über die Verteilung der Zahlen, die zu den $n$ ersten Primzahlen teilerfremd sind. *Comm. Phys. Math. Helsingfors* 25 (1931), 1–37.

[6] Wright, E.M.: A prime-representing function. *Amer. Math. Monthly* 58 (1951), 616–618.

Bakir Farhi
Institut des Hautes Études Scientifiques
Le Bois-Marie
35, route de Chartres
F–91440 Bures-sur Yvette, France
e-mail: bakir.farhi@gmail.com