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Isoperimetric characterization of the incenter of a triangle

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1 Introduction

Recently Katsuyuki Shibata introduced a new kind of center of a triangle, which he calls the *illuminating center* ([3]). It is a point that maximizes the total brightness of a triangular park Ω obtained by a light source on that point, namely, a point that maximizes $V_0(x) = \int_\Omega |x-y|^{-2}\,d\mu(y)$, where μ is the standard Lesbegue measure of $\mathbb{R}^2$. Unfortunately, $V_0(x)$ is not well-defined; it diverges for any point in Ω. In order to produce a well-defined potential, Shibata used the cut-off of the divergence of the integrand.

In [2] the author introduced the renormalization of $\int_\Omega |x-y|^{\alpha-m}\,d\mu(y)$ (which is called the Riesz potential when $0<\alpha<m$) of a compact subset Ω in $\mathbb{R}^m$ which is a closure of an open set for $\alpha \le 0$ to obtain a one-parameter family of (*renormalized*) *potentials* $V_\Omega^{(\alpha)}$, and studied the points where the extremal values of $V_\Omega^{(\alpha)}$ are attained, which we call the $r^{\alpha-m}$*-centers* of Ω. The notion of $r^{\alpha-m}$-centers includes not only Shibata's illuminating center of a planar domain as an r^{-2}-center, but also the center of mass of any compact set $\Omega \subset \mathbb{R}^m$ as r^2-center. This is because the center of mass x_G is given by

Clark Kimberling listet auf seiner Web-Seite *Encyclopedia of Triangle Centers* inzwischen weit über 5000 Dreieckszentren auf. Dort ist z.B. $X(1)$ der Inkreismittelpunkt, $X(2)$ der Schwerpunkt, oder $X(54)$ der Kosnita-Punkt eines Dreiecks. Zahlreiche dieser Zentren lassen sich auf unterschiedliche Weise charakterisieren. In der vorliegenden Arbeit wird gezeigt, dass der Inkreismittelpunkt gleichzeitig eine gewisse Funktion minimiert: Dazu betrachtet man das Dreieck als Grundfläche einer Pyramide mit Spitze p. Aus deren Volumen und Oberfläche bildet man sodann einen geeigneten skaleninvarianten von p abhängigen Quotienten. Minimiert man die so definierte Funktion so fällt die Projektion des optimalen Punktes p auf die Grundfläche just in den Inkreismittelpunkt des Dreicks.

$x_G = \int_\Omega y\, d\mu(y) / \int_\Omega 1\, d\mu(y)$, or equivalently by $\int_\Omega (x_G - y)\, d\mu(y) = 0$, which implies that it can be characterized as a unique critical point of the map $V_\Omega^{(m+2)} : \mathbb{R}^m \ni x \mapsto \int_\Omega |x - y|^2\, d\mu(y) \in \mathbb{R}$.

Shibata announced[1] a theorem that an r^a-center of a non-obtuse triangle approaches the circumcenter as a goes to $+\infty$ and to the incenter as a goes to $-\infty$. The proof with more generality is given in [2]. Thus, we can give interpretations of the barycenter, circumcenter, and incenter of a triangle as points that optimize a kind of potential and the limits of them.

The motivation of the theorem in this note comes from the same philosophy; to express a center as a point that optimizes a kind of potential. Our potential in this note is the ratio of the volume of the cone over a given triangle Ω and the area of its boundary, with the former being squared and the latter cubed to make the ratio scale invariant. Then, the image of the regular projection of a vertex of a cone that optimizes this ratio is nothing but the incenter.

2 Cone isoperimetric center

Let Ω be a compact set which is a closure of an open subset of $\mathbb{R}^2$ with a piecewise C^1 boundary $\partial\Omega$. We assume that $\mathbb{R}^2$ is embedded in $\mathbb{R}^3$ in a standard way; $\mathbb{R}^2 = \{(x_1, x_2, 0) \in \mathbb{R}^3 \mid x_i \in \mathbb{R}\}$. Let Π_h denote a level plane in $\mathbb{R}^3$ with height $h > 0$, $\Pi_h = \{x_3 = h\}$, and C_p a cone over Ω with vertex $p \in \Pi_h$, $C_p = \{tx + (1-t)p \mid x \in \Omega,\ 0 \le t \le 1\}$. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the regular projection.

Definition 2.1.

(1) Let p_h be a point in Π_h where the minimum value of a function $\Pi_h \ni p \mapsto \mathrm{Area}\,(\partial C_p)$ is attained. We call $\pi(p_h)$ a *cone isoperimetric center of Ω of height h*.

(2) Let p be a point in $\mathbb{R}^3_+ = \{x_3 > 0\}$ that gives the minimum value of a function

$$f(p) = \frac{\big(\mathrm{Area}\,(\partial C_p)\big)^3}{\big(\mathrm{Vol}\,(C_p)\big)^2}.$$

We call C_p an *isoperimetrically optimal cone* and $\pi(p)$ a *cone isoperimetric center of Ω*.

Lemma 2.2. *Let $\triangle ABC$ be a triangle. Then there exists a cone isoperimetric center of height h for any $h > 0$.*

Proof. Let S be the area, and a, b, and c the lengths of the edges BC, CA, and AB, respectively. Fix $h > 0$. Let $P \in \Pi_h$ be a point and $D = \pi(P)$. Let u, v, and w be the distances with signs between D and the lines $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. The signs of u, v, and w are given as follows. We put $u > 0$ if D and A are in the same half-plane cut out by the line $\overline{BC}$. Remark that the position of D is determined uniquely by u and v.

1. at 2010 Autumn Meetings of the Mathematical Society of Japan

Then the area of the triangle ΔABC is given by $S = \frac{1}{2}(au + bv + cw)$, and the area of the boundary of the cone is given by

$$\text{Area}\,(\partial C_P) = S + \frac{1}{2}\left(a\sqrt{u^2+h^2} + b\sqrt{v^2+h^2} + c\sqrt{w^2+h^2}\right). \tag{1}$$

Let the right-hand side of (1) be denoted by $\psi(D)$. Then, it takes the value $S + \frac{1}{2}(a+b+c)\sqrt{r^2+h^2}$ at the incenter I, where r is the radius of the inscribed circle. Put

$$\rho = \frac{a+b+c}{\min\{a,b,c\}}\sqrt{r^2+h^2}.$$

Let $\bar{N}_\rho(\overline{BC})$ be the set of points so that the distance to the line $\overline{BC}$ is not greater than ρ, namely, a closed strip with central axis $\overline{BC}$ which is 2ρ wide. Two other strips, $\bar{N}_\rho(\overline{CA})$ and $\bar{N}_\rho(\overline{AB})$, can be defined similarly. Put $K = \bar{N}_\rho(\overline{BC}) \cap \bar{N}_\rho(\overline{CA}) \cap \bar{N}_\rho(\overline{AB})$. Then K is a compact set containing I.

Suppose $D \notin K$. Then at least one of $|u|$, $|v|$, and $|w|$ is greater than ρ. Therefore,

$$\psi(D) > S + \frac{1}{2}\min\{a,b,c\}\sqrt{\rho^2+h^2} > S + \frac{1}{2}\min\{a,b,c\}\rho = \psi(I),$$

which implies $\inf_{D'\in K}\psi(D') = \inf_{D''\in\mathbb{R}^2}\psi(D'')$. Since ψ is continuous and K is compact, there is a point $D \in K$ where $\inf_{D'\in K}\psi(D')$ is attained.

It follows that $\inf_{D''\in\mathbb{R}^2}\psi(D'')$ is also attained at D. □

Theorem 2.3. *Let $\triangle ABC$ be a triangle. The cone isoperimetric center of height h coincides with the incenter for any $h > 0$. The height of the isoperimetrically optimal cone is $2\sqrt{2}$ times the radius of the inscribed circle.*

Proof. (1) Let us use the same notation as in Lemma 2.2.

Let D_h be a cone isoperimetric center of ΔABC of height h, and u_h, v_h, and w_h be the signed distances between D_h and the lines $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Then the pair (u_h, v_h) minimizes a function

$$F(u,v) = a\sqrt{u^2+h^2} + b\sqrt{v^2+h^2} + c\sqrt{\left(\frac{2S-au-bv}{c}\right)^2 + h^2}.$$

Therefore, when $(u,v,w) = (u_h, v_h, w_h)$ we have

$$F_u(u,v) = \frac{au}{\sqrt{u^2+h^2}} + \frac{cw}{\sqrt{w^2+h^2}}\cdot\left(-\frac{a}{c}\right) = 0,$$
$$F_v(u,v) = \frac{bv}{\sqrt{v^2+h^2}} + \frac{cw}{\sqrt{w^2+h^2}}\cdot\left(-\frac{b}{c}\right) = 0,$$

which implies

$$\frac{u}{\sqrt{u^2+h^2}} = \frac{v}{\sqrt{v^2+h^2}} = \frac{w}{\sqrt{w^2+h^2}}. \tag{2}$$

Remark that the above holds only when u, v, and w are all positive, implying that D_h is in the interior of ΔABC. The equation (2) means that three angles between the xy-plane and three planes through PAB, PBC, and PCA are all equal. Therefore, each pair of the three planes is symmetric in a plane which is orthogonal to the xy-plane and contains the intersection line of the pair. These three symmetries show that the three lines D_hA, D_hB, and D_hC, which are the intersections of the xy-plane and the three planes of the symmetries, are the angle bisectors of $\angle A$, $\angle B$, and $\angle C$, respectively. It follows that D_h coincides with the incenter of ΔABC.

(2) The second statement follows from elementary calculus. Let r be the radius of the inscribed circle. Put $P_h = \pi^{-1}(D_h) \cap \Pi_h$, then

$$\text{Area}\,(\partial C_{P_h}) = S + \frac{1}{2}(a+b+c)r\sqrt{1+\left(\frac{h}{r}\right)^2} = S\left(1+\sqrt{1+\left(\frac{h}{r}\right)^2}\right).$$

As $\text{Vol}\,(C_{P_h}) = \frac{1}{3}Sh$,

$$f(P_h) = \frac{\left(\text{Area}\,(\partial C_{P_h})\right)^3}{\left(\text{Vol}\,(C_{P_h})\right)^2} = 9S\frac{\left(1+\sqrt{1+\left(\frac{h}{r}\right)^2}\right)^3}{h^2} = \frac{9S}{r^2}\cdot\frac{\left(1+\sqrt{1+\left(\frac{h}{r}\right)^2}\right)^3}{\left(\frac{h}{r}\right)^2}.$$

Since $\varphi(t) = \frac{\left(1+\sqrt{1+t^2}\right)^3}{t^2}$ $(t > 0)$ takes the minimum at $t = 2\sqrt{2}$, it completes the proof. $\square$

Remark 2.4. The above theorem means that the cone isoperimetric center of height h is identically the same for any $h > 0$ and that it coincides with the limit of r^a-center as a goes to $-\infty$ for triangles. But it does not hold in general as an example below shows.

Let us call a point an *asymptotic $r^{-\infty}$-center* of Ω if it is the limit of a convergent sequence of r^{a_i}-centers with $a_i \to -\infty$ as $i \to +\infty$. We showed in [2] that an asymptotic $r^{-\infty}$-center is a *max-min point* of Ω, by which we mean a point that gives the supremum of a map $\mathbb{R}^2 \ni x \mapsto \min_{y\in\overline{\Omega^c}} |y-x| \in \mathbb{R}$, where $\overline{\Omega^c}$ denotes the closure of the complement of Ω. We remark that an r^a-center $(a \le -2)$ and a max-min point are not necessarily unique. To see this, it is enough to consider a disjoint union of two rectangles, say, $\Omega' = \{(\xi, \eta) \mid 1 \le |\xi| \le 2,\ |\eta| \le 2\}$.

Let Ω be a trapezoid given by $\Omega = \{(\xi, \eta) \mid 0 \le \xi \le 2, |\eta| \le 1 + \frac{1}{2}\xi\}$. It is easy to see that a cone isoperimetric center of height h is on the ξ-axis for any h. Let it be given by $(\xi_h, 0)$. Numerical experiments show that $\xi_1 \sim 0.9169$, $\xi_2 \sim 0.9079$, $\xi_3 \sim 0.9045$, and $\xi_4 \sim 0.9031$, and the minimum of the ratio f is attained at $h \sim 3.250$ when $\xi_h \sim 0.90405$. On the other hand, an asymptotic $r^{-\infty}$-center is $(1, 0)$. This is because the set of max-min points is $\{(1, \eta) \mid |\eta| \le \frac{3}{2} - \frac{\sqrt{5}}{2}\}$ whereas any r^a-center is contained in $\{(\xi, 0) \mid 1 \le \xi \le \frac{7}{4}\}$ for any a by the symmetry argument (based on the moving plane method [1]) explained in [2], and the point $(1, 0)$ is the unique intersection point of these sets.

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