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Geometric median in the plane

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1 Introduction

Let $T_i = (x_1^{(i)}, x_2^{(i)})$, $i = 1, \dots, m$, $m \geq 2$, be a given set of the points in the plane, with their corresponding weights $w_i > 0$. We need to determine the point $T = (u, v) \in \mathbb{R}^2$ such that the weighted sum of Euclidean distances between the points T_i and T be minimal, i.e., we need to minimize the functional $F : \mathbb{R}^2 \to \mathbb{R}$,

$$F(u_1, u_2) = \sum_{i=1}^{m} w_i\, d\big(T_i(x_1^{(i)}, x_2^{(i)}), T(u_1, u_2)\big) = \sum_{i=1}^{m} w_i \sqrt{\big(x_1^{(i)} - u_1\big)^2 + \big(x_2^{(i)} - u_2\big)^2}\,.$$

The point T with the above property is called *the weighted geometric median* of points $T_i = (x_1^{(i)}, x_2^{(i)})$, $i = 1, \dots, m$.

This problem and its generalizations often occur in a variety of applications, such as determining the location of schools, medical emergency centers, fire stations, bus stations or garages, telecommunication centers, etc. (see [4], [5]).

In the scientific literature (for example, see [11]), it is considered that Pierre de Fermat (1601–1665) first started to deal with this problem by considering the problem of determining the geometric median of three points in the plane. The Italian mathematician

Ein bekanntes Problem der Elementargeometrie besteht darin, in der Ebene denjenigen Punkt zu finden, welcher die Abstandssumme zu drei gegebenen Punkten minimiert. Bereits Pierre de Fermat und Evangelista Torricelli beschäftigten sich mit dieser Fragestellung. Verallgemeinernd kann man den geometrischen Median bei m Punkten im n-dimensionalen Raum für eine gewichtete Abstandssumme untersuchen. Die Autoren befassen sich in der vorliegenden Arbeit just mit diesem Problem und wenden dabei ihre Aufmerksamkeit auch dem Weiszfeld-Algorithmus zur Bestimmung der Lösung zu.

Evangelista Torricelli (1608–1647) also considered this problem, hence the geometric median is sometimes called *Torricelli point*. This problem was also addressed by the Italian mathematician Battista Cavalieri (1598–1647), the English mathematician Thomas Simpson (1710–1761), etc. The problem became interesting again in the twentieth century when it was realized that it lies in the background of many practical problems. The Hungarian mathematician Endre Vaszonyi Weiszfeld is of particular interest as he also defined the first numerical iterative algorithm for finding the geometric median for a set of points in a 3-space in 1936 (see [13]). Amending some of Weiszfeld's arguments, Kuhn ([9], [10]) proved in 1962 that the optimal solution is at one of the given points, but such a claim was valid only with some additional hypotheses. Drezner (see [2], [3]) constructed Weiszfeld's accelerated algorithm, and in 1974 Katz (see [8]) showed that in general, the convergence of Weiszfeld's algorithm is linear.

2 Determining the geometric median of three non-collinear points in the plane

Let $A, B, C \in \mathbb{R}^2$ be three non-collinear points in the plane which define the triangle ABC. A term of an oriented angle, which we need for proving the basic theorem for determining the geometric median of the triangle ABC, is specified below. Specifically, if the points $A, B, C \in \mathbb{R}^2$ are collinear, the geometric median is any point in the convex hull of these points.

Definition 1. An oriented angle, which is formed by the lines l_1 and l_2 and denoted by $\measuredangle(l_1, l_2)$, is an angle for which we need to rotate the line l_1 in positive orientation so that it coincides with the line l_2 or it is parallel to the line l_2.

An oriented angle $\measuredangle(BA, BC)$ is an angle which is formed by the lines AB and BC, or an angle for which we need to rotate the line AB around the point B in positive orientation so that it coincides with the line BC.

Remark 1. The size of an oriented angle $\measuredangle(BA, BC)$ can be equal to $\angle ABC$ or to its supplement.

If the triangle ABC is positively oriented, it is easy to see that the oriented angles $\measuredangle(BA, BC)$, $\measuredangle(CB, CA)$, $\measuredangle(AC, AB)$ are equal to the corresponding outer angles of the triangle, while the inner angles are $\measuredangle(BC, BA)$, $\measuredangle(CA, CB)$, $\measuredangle(AB, AC)$.

Lemma 1. *The points P, Q, R, S are concyclic if and only if* $\measuredangle(PR, PS) = \measuredangle(QR, QS)$.

The proof of Lemma 1 is contained in [7].

Remark 2. There are well-known claims that inscribed angles of the same circular arc are equal, and that the opposite angles of a convex quadrangle are supplementary if and only if its angles are concyclic points. These two claims can be consolidated into one theorem. Namely, if four points P, Q, R, S lie on one circle, then $\angle RPS$ and $\angle RQS$ are equal or supplementary, depending on wether the points P and Q are lying on the same side or on opposite sides with respect to the RS and vice versa.

Let us note that, if P and Q are in the same half-plane with respect to RS, then

$$\measuredangle(PR, PS) = \measuredangle(QR, QS),$$

and vice versa. However, if P and Q are in different half-planes with respect to RS, then $\angle RPS$ and $\angle RQS$ are supplementary, but $\measuredangle(PR, PS) = \measuredangle(QR, QS)$, and vice versa. The equality $\measuredangle(PR, PS) = \measuredangle(QR, QS)$ is a necessary and sufficient condition that the four points lie on the circle.

Theorem 1. *If ABC', BCA', CAB' are equilateral triangles on the outer side of a given triangle ABC, then the lines AA', BB', CC' and the circles circumscribed around the triangles ABC', BCA', CAB' intersect at one point $T \in \mathbb{R}^2$. In addition,*

$$d(A, A') = d(B, B') = d(C, C'),$$

and the lines CC', BB', AA' form angles of $60°$ (*Figure* 1).

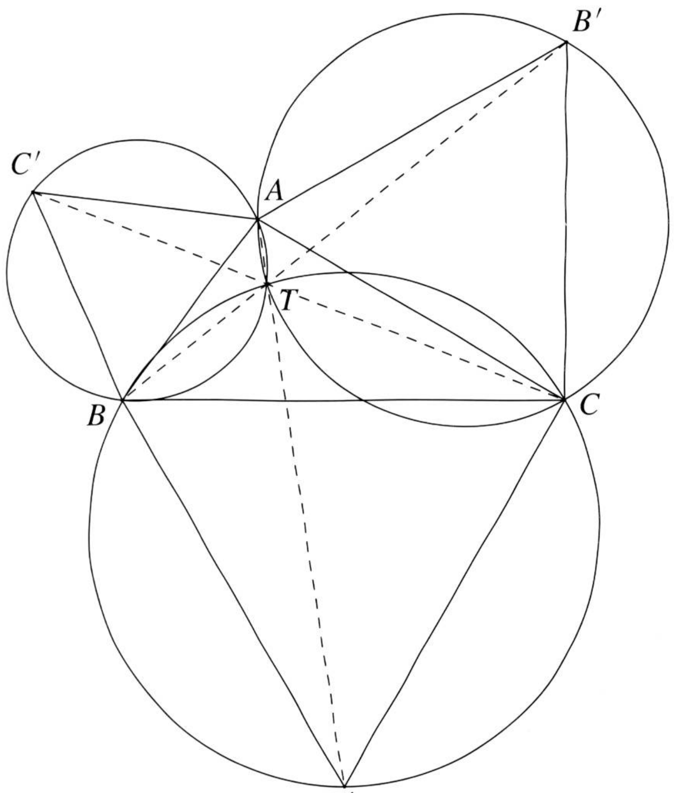

Fig. 1

Proof. Assume that the triangle ABC is positively oriented. Then the triangles ABC', BCA', CAB' are negatively oriented. Rotation around the point A by $60°$ maps the point C' to B, and the point C to B'. This implies

$$d(C, C') = d(B, B') \quad \text{and} \quad \measuredangle(CC', BB') = 60°.$$

Analogously,

$$d(A, A') = d(C, C') \quad \text{and} \quad \measuredangle(AA', CC') = 60^\circ,$$
$$d(B, B') = d(A, A') \quad \text{and} \quad \measuredangle(BB', AA') = 60^\circ.$$

Assume that $BB' \cap CC' = T$. As $\measuredangle(TC', TB) = 60^\circ = \measuredangle(AC', AB)$, and according to Lemma 1, T lies on the circle around the triangle ABC', from which follows

$$\measuredangle(TB, TA) = 60^\circ = \measuredangle(C'B, C'A). \tag{1}$$

As

$$\measuredangle(TC, TB') = 60^\circ = \measuredangle(A'C, A'B),$$

according to Lemma 1, T lies on the circle BCA', from which follows

$$\measuredangle(TB, TA') = \measuredangle(CB, CA') = 60^\circ. \tag{2}$$

From (1) and (2) we can see that $\measuredangle(TB, TA) = \measuredangle(TB, TA')$, from which it follows that the lines TA and TA' are identical, i.e., the points A, A', T are collinear. □

The point T from the previous theorem is called Torricelli point of the triangle ABC, and the lines AA', BB' and CC' are called Simpson lines.

Corollary 1. *Let $A, B, C \in \mathbb{R}^2$ be three non-collinear points in the plane.*

(i) *If $\triangle ABC$ has no angle greater than 120°, then the Torricelli point S lies within $\triangle ABC$.*

(ii) *If $\triangle ABC$ has an angle equal to 120°, then the Torricelli point S is the vertex at that angle.*

(iii) *If $\triangle ABC$ has an angle greater than 120°, then the Torricelli point S lies outside $\triangle ABC$.*

Proof. Let $\alpha = \angle BAC$ (Figure 2). Since $\triangle ABC'$, $\triangle BCA'$, and $\triangle CAB'$ are equilateral triangles, we have $\angle BAC' = \angle CAB' = 60^\circ$.

(i) If $\alpha + 60^\circ < 180^\circ$, then the lines BB' and CC' intersect at the vertex T within $\triangle ABC$ (Figure 2a)).

(ii) If $\alpha + 60^\circ = 180^\circ$, then the lines BB' and CC' intersect at the vertex A (Figure 2b)).

(iii) If $\alpha + 60^\circ > 180^\circ$, then the lines BB' and CC' intersect at the point T outside $\triangle ABC$ (Figure 2c)).

By Theorem 1, the point $T = BB' \cap CC'$ lies on the line AA' which corresponds to the Torricelli point. □

Theorem 2. *The geometric median of three non-collinear points $A, B, C \in \mathbb{R}^2$ is located within the triangle ABC.*

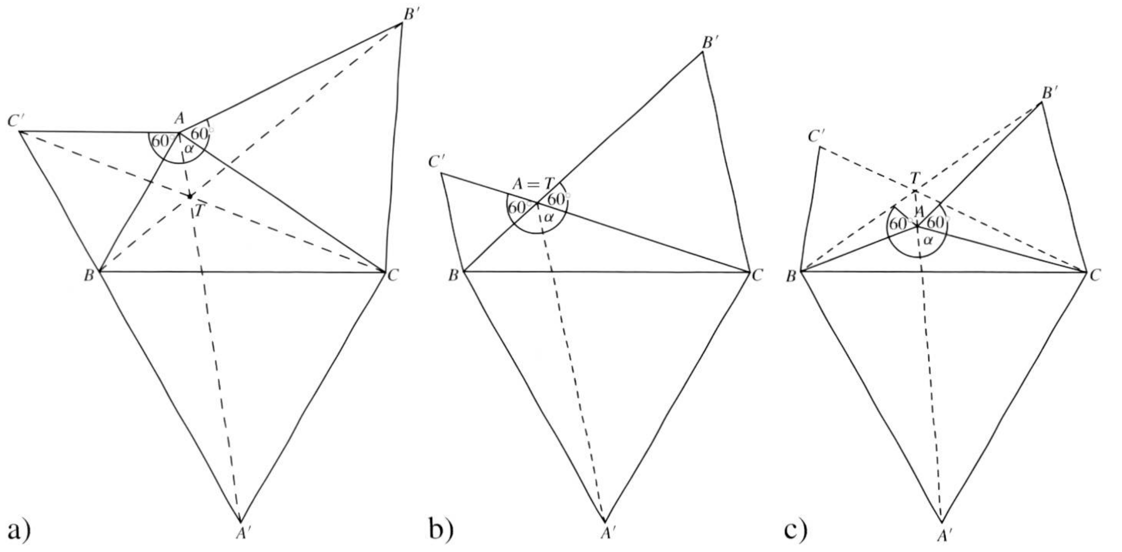

Fig. 2

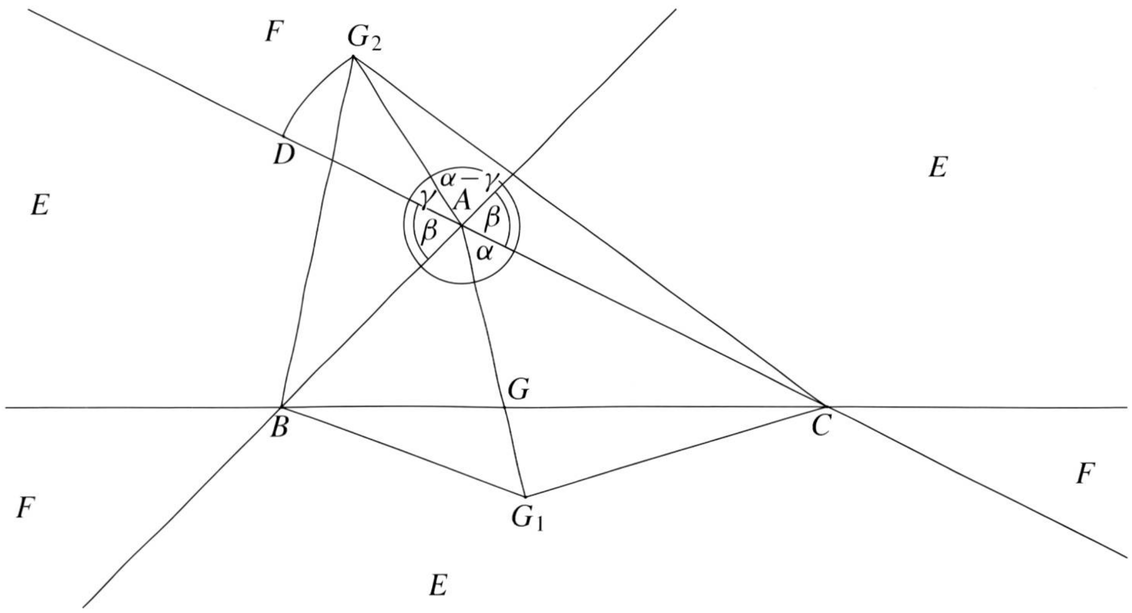

Fig. 3

Proof. We will show that for any given point outside $\triangle ABC$ there exists a point G on one of the edges of that triangle such that the sum of distances from G to vertices of $\triangle ABC$ does not exceed the analogous sum for the given point.

Look at Figure 3. We distinguish two cases:

1. Choose an arbitrary point G_1 outside $\triangle ABC$, in the area E, and let $G = BC \cap AG_1$. We will prove that

$$d(G, A) + d(G, B) + d(G, C) \leq d(G_1, A) + d(G_1, B) + d(G_1, C). \qquad (3)$$

From the triangle $\triangle BG_1C$ we see that

$$d(G, B) + d(G, C) = d(B, C) \leq d(G_1, B) + d(G_1, C), \tag{4}$$

and obviously

$$d(A, G) \leq d(G_1, A). \tag{5}$$

Adding (4) and (5) gives (3).

2. Let G_2 be an arbitrary point outside $\triangle ABC$ in the area F. We will prove that

$$d(A, A) + d(A, B) + d(A, C) \leq d(G_2, A) + d(G_2, B) + d(G_2, C). \tag{6}$$

Depending on the angle $\gamma = \angle DAG_2$ we have two cases:

i) If $\gamma + \beta \geq (\alpha - \gamma) + \beta$, then $\angle BAG_2 = \gamma + \beta$ is the greatest angle in $\triangle BAG_2$, hence

$$d(A, B) \leq d(G_2, B). \tag{7}$$

On the other hand, from $\triangle ACG_2$ we see that

$$d(A, C) \leq d(A, G_2) + d(G_2, C). \tag{8}$$

Adding (7) and (8) gives (6).

ii) If $\gamma + \beta \leq (\alpha - \gamma) + \beta$, then $\angle CAG_2 = (\alpha - \gamma) + \beta$ is the greatest angle in $\triangle CAG_2$, hence

$$d(A, C) \leq d(G_2, C). \tag{9}$$

From $\triangle ABG_2$ we have

$$d(A, B) \leq d(A, G_2) + d(G_2, B), \tag{10}$$

and adding (9) and (10) again gives (6). □

Theorem 3. *Let $A, B, C \in \mathbb{R}^2$ be three non-collinear points in the plane.*

(i) *If $\triangle ABC$ has no angle greater than $120°$, then the geometric median of points A, B, and C agrees with the Torricelli point.*

(ii) *If $\triangle ABC$ has an angle greater than $120°$, then the geometric median of points A, B, and C is located at the vertex corresponding to that angle.*

Proof. (i) Assume that $\triangle ABC$ has no angle greater than $120°$. Let P be an arbitrary point within $\triangle ABC$ (Figure 4a). We will show that the following holds:

$$d(P, A) + d(P, B) + d(P, C) \geq d(T, A) + d(T, B) + d(T, C).$$

Let $P' \in \mathbb{R}^2$ be a point such that $\triangle CPP'$ is an equilateral triangle. Rotation around C by $-60°$ transforms $\triangle APC$ onto $\triangle B'P'C$, so $\triangle APC \cong \triangle B'P'C$. Therefore

$$d(P, A) + d(P, C) + d(P, B) = d(B', P') + d(P', P) + d(P, B) \geq d(B, B'), \tag{11}$$

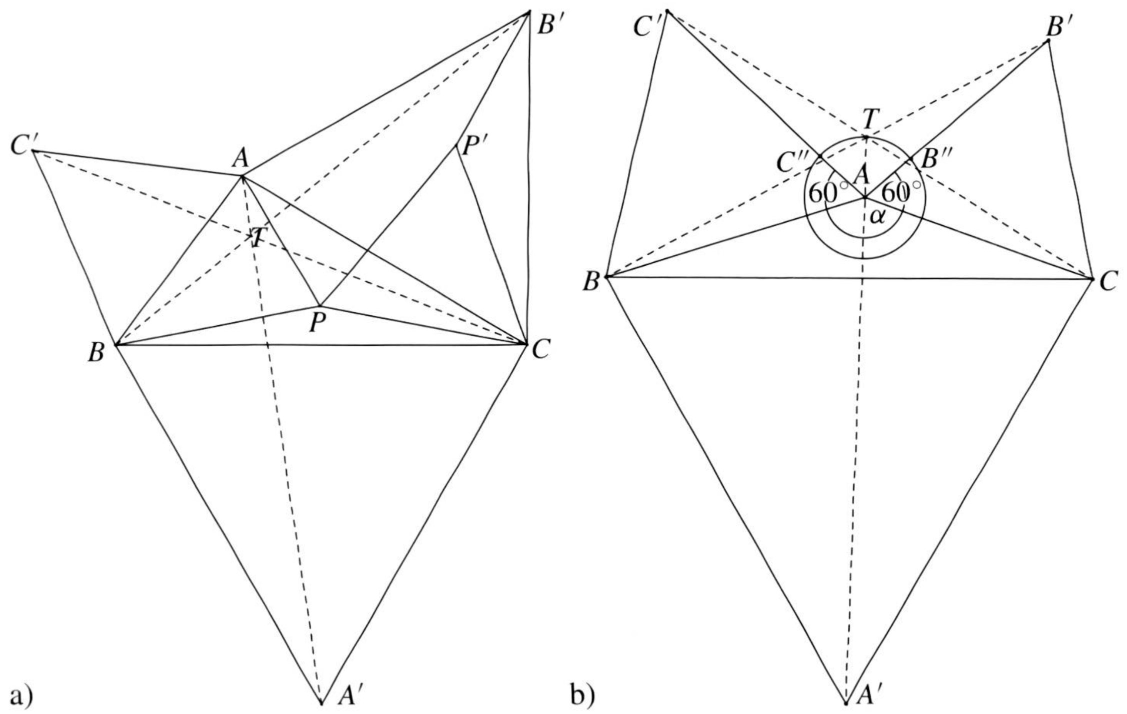

Fig. 4

From $\triangle A'CA \cong \triangle BCB'$ and $\triangle ABA' \cong \triangle CBC'$ it is easy to see that

$$d(A, T) + d(B, T) + d(C, T) = d(B, B'). \tag{12}$$

(11) and (12) gives

$$d(P, A) + d(P, B) + d(P, C) \geq d(A, T) + d(B, T) + d(C, T).$$

(ii) Assume now that one of the angles of $\triangle ABC$, say the one at A, is greater than $120°$ (Figure 4b)),

$$120° < \angle BAC < 180°. \tag{13}$$

Simpson lines intersect in the point T which does not belong to $\triangle ABC$. We show the following

$$d(A, A) + d(A, B) + d(A, C) \leq d(T, A) + d(T, B) + d(T, C), \tag{14}$$

and for all $T_1 \in \triangle ABC$

$$d(A, A) + d(A, B) + d(A, C) \leq d(T_1, A) + d(T_1, B) + d(T_1, C). \tag{15}$$

Firstly, let us prove (14). From $\angle BAC' = \angle CAB' = 60°$ and (13) it follows that $60° < \angle C'AB' < 120°$. Two cases are possible:

a) $\angle TAC' > 30°$
In this case, $\angle BAT > 90°$ is the greatest angle in $\triangle BAT$, so we have

$$d(B, T) \geq d(A, B). \tag{16}$$

Using the triangle inequality $d(A, C) \leq d(A, T) + d(T, C)$ and (16) we get (14).

b) $\angle TAB' > 30°$

In this case, $\angle CAT > 90°$ is the greatest angle in $\triangle CAT$, so

$$d(C, T) \geq d(A, C). \tag{17}$$

Using $d(A, B) \leq d(A, T) + d(T, B)$ and (17) we obtain (14).

All that remains is to prove (15). Let T_1 be an arbitrary point in $\triangle ABC$ and let $T_2 \in \mathbb{R}^2$ be such that $\triangle BT_1T_2$ is an equilateral triangle (Figure 5).

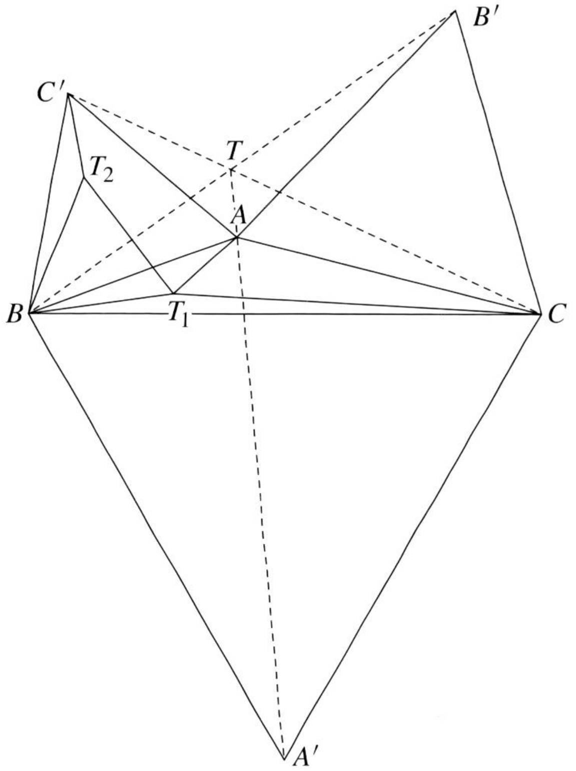

Fig. 5

Rotating $\triangle BAT_1$ by $-60°$ around B, we get $\triangle BC'T_2$. Therefore $\triangle BAT_1 \cong \triangle BC'T_2$, hence

$$d(T_1, A) + d(T_1, B) + d(T_1, C) = d(T_1, C) + d(T_1, T_2) + d(T_2, C').$$

It is easy to see that

$$d(A, C') + d(A, C) \leq d(C', T_2) + d(T_2, T_1) + d(T_1, C),$$

i.e.,

$$d(A, B) + d(A, C) \leq d(T_1, A) + d(T_1, B) + d(T_1, C). \qquad \square$$

3 Geometric median of *m* points in the plane

In the previous chapters we considered the problem of finding the geometric median of three non-collinear points in the plane. Next, we will consider the problem of determining the geometric median of finitely many points in a finite-dimensional space.

3.1 Weiszfeld's algorithm

Let $T_i = (x_1^{(i)}, \dots, x_n^{(i)})$, $i = 1, \dots, m$, $m \geq 2$, be a given set of points in $\mathbb{R}^n$, with their corresponding weights $w_i > 0$. We have to find a point $T^* = (u_1^*, \dots, u_n^*) \in \mathbb{R}^n$ such that the sum of weighted Euclidean distances from these points to T^* is minimal. This problem reduces to the problem of minimization for the functional $F\colon \mathbb{R}^n \to \mathbb{R}$ given by

$$\begin{aligned} F(u_1, \dots, u_n) &= \sum_{i=1}^{m} w_i \rho_i(u_1, \dots, u_n), \\ \rho_i(u_1, \dots, u_n) &= \sqrt{\sum_{j=1}^{n} (x_j^{(i)} - u_j)^2}, \quad i = 1, \dots, m. \end{aligned} \tag{18}$$

The following lemma lists some properties of the functional F (see, e.g., [1]).

Lemma 2. *Let $T_i = (x_1^{(i)}, \dots, x_n^{(i)}) \in \mathbb{R}^n$, $i = 1, \dots, m$, $m \geq 2$, be a given set of points, with their corresponding weights $w_i > 0$, and let $F\colon \mathbb{R}^n \to \mathbb{R}$ be as in* (18). *Then*

(i) *F is continuous.*

(ii) *F is convex.*

(iii) *There exists a point $T^* = (u_1^*, \dots, u_n^*) \in \mathbb{R}^n$ at which F attains its global minimum.*

As one of the methods for finding the geometric median, we are going to briefly describe Weiszfeld's iterative procedure for determining the global minimum of the functional F, which is highly regarded in applications (see [5]). To simplify the notations, we consider the case $n = 2$. So, we are given the points $T_1 = (x_1^{(1)}, x_2^{(1)}), \dots, T_m = (x_1^{(m)}, x_2^{(m)})$ and their respective non-negative weights $w_1, \dots, w_m$, and we have to find a point $T^* = (u_1^*, u_2^*) \in \mathbb{R}^2$ at which the functional F attains its global minimum. Equating the gradient of F to zero, we get the following system of equations

$$\frac{\partial F(u_1, u_2)}{\partial u_1} = \sum_{i=1}^{m} \frac{w_i (u_1 - x_1^{(i)})}{\rho_i(u_1, u_2)} = 0, \qquad \frac{\partial F(u_1, u_2)}{\partial u_2} = \sum_{i=1}^{m} \frac{w_i (u_2 - x_2^{(i)})}{\rho_i(u_1, u_2)} = 0. \tag{19}$$

(Obviously, the partial derivatives do not exist at $T_1, \dots, T_m$.)

In general, system (19) cannot be solved explicitly for an $m > 3$. If we write it in the form

$$u_1 = \varphi(u_1, u_2), \qquad u_2 = \psi(u_1, u_2), \tag{20}$$

$$u_1 = \frac{1}{\sum_{i=1}^{m} \frac{w_i}{\rho_i(u_1, u_2)}} \sum_{i=1}^{m} \frac{w_i x_1^{(i)}}{\rho_i(u_1, u_2)}, \qquad u_2 = \frac{1}{\sum_{i=1}^{m} \frac{w_i}{\rho_i(u_1, u_2)}} \sum_{i=1}^{m} \frac{w_i x_2^{(i)}}{\rho_i(u_1, u_2)},$$

then, according to the method of simple iterations (see [6]), we can define Weiszfeld's iterative process

$$u_1^{(k+1)} = \varphi(u_1^{(k)}, u_2^{(k)}), \qquad u_2^{(k+1)} = \psi(u_1^{(k)}, u_2^{(k)}), \quad k = 0, 1, \dots \tag{21}$$

As an initial approximation, we can take, for instance, the centroid of the given points $T_1, \dots, T_m$

$$u_1^{(0)} = \frac{1}{W}\sum_{i=1}^{m} w_i x_1^{(i)}, \qquad u_2^{(0)} = \frac{1}{W}\sum_{i=1}^{m} w_i x_2^{(i)}, \qquad W = \sum_{i=1}^{m} w_i. \tag{22}$$

4 Examples

We will present three examples illustrating the properties and applications of the geometric median in the plane. Examples 1 and 2 illustrate the difference between the geometric median and the centroid for three non-collinear points A, B, and C, depending on the distances between these points. In Example 3, we give a problem of location, in which, by applying Weiszfeld's algorithm, we find the optimal solution.

Example 1. Take the points $A(5.00, 4.21)$, $B(2.96, 1.90)$, $C(7.04, 1.90)$ in the plane. The mutual distances between them are not significantly different (see Figure 6a)), and thus the Torricelli point T is inside $\triangle ABC$ and it coincides with the geometric median of these points. The centroid of these points is $D(5.00, 2.67)$ and $d(T, D) = 0.41$.

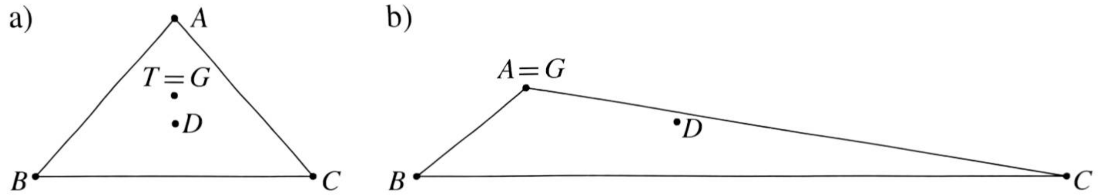

Fig. 6

Example 2. For the next example take the points $A(2.80, 3.17)$, $B(1.19, 1.88)$, $C(10.74, 1.88)$ which are such that $\triangle ABC$ has one of the angles greater than $120°$ (see Figure 6b)). The Torricelli point T is now located outside $\triangle ABC$ and it does not coincide with the geometric median G of the given points A, B, C. The centroid of these points $D(4.91, 2.31)$ is quite distant from the geometric median: $d(G, D) = 2.26$.

Example 3. (See [12].) A fire station needs to be built in one region of the State of Massachusetts so that a fire-fighting vehicle, arrives in a maximum of six minutes from the time it receives a call to the place of fire. It is presumed that the call requires one minute, and the same amount of time is required until fire-fighters are ready to go.

Let us observe 15 settlements, whose position in the coordinate system is determined by the points $T_i = (x_i, y_i)$, with corresponding weights $w_i = 1, i = 1, \dots, 15$ (see Table 1). If we assume that the average speed of a fire-fighting vehicle is 100 km/h, meaning that for the remaining 4 minutes the vehicle can travel 6.7 km.

We want to determine a point G which will represent a fire station so that the sum of distances from that point to points $T_i = (x_i, y_i)$, $i = 1, \dots, 15$, is minimal. The process is performed by using Weiszfeld's algorithm. As an initial approximation, we take

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
x_i	13.24	9	14	2	4	11.56	8	17	1	6	9	13	6	5	14
y_i	−4.13	−6	−5	6	1	−2.86	−1	−4	−4	−3	1	1.15	−2	4.35	−3

Table 1

the centroid of the provided data which are calculated by using formula (22), which is $D(8.85333, -1.43267)$. We follow an iterative process described by formulas (19)–(21). We observe the required number of iterations for which the norm of differences between every two successive approximations of the solution G would be less than some predefined precision ϵ. We can see that, with the increasing precision, the number of iterations increases linearly (see Table 2), which confirms the theoretical result mentioned in the introduction, which states that the convergence of Weiszfeld's algorithm is linear.

Precision (ϵ)	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}
Number of iteration	3	9	16	23	30	38	45	52

Table 2

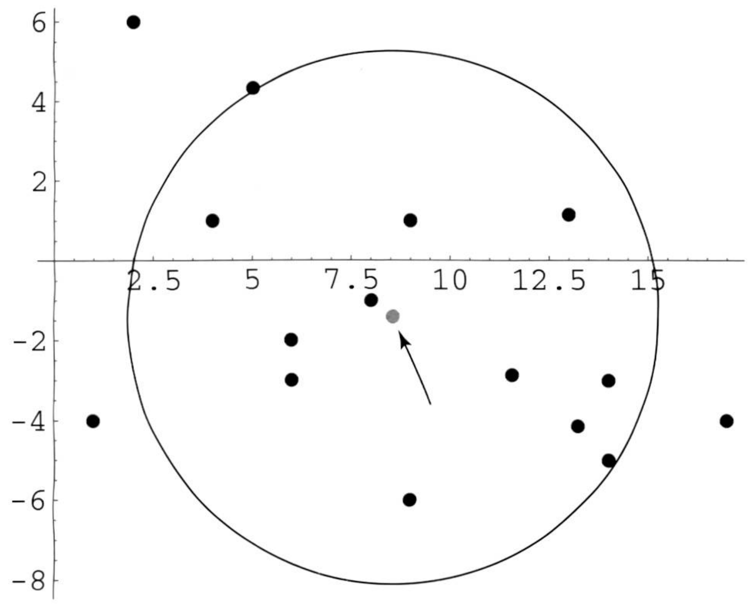

Fig. 7

In Figure 7, settlements are represented by black points.

The grey point $G = (8.56372, -1.40877)$ (which the arrow points to) is the geometric median, which determines the position of the fire station and which was determined by Weiszfeld's algorithm after 52 iterations. If we plot a circle of radius 6.7 km centered at the geometric median, we can see that some of the settlements are not within the circle; hence, they cannot be well covered by the fire station. We conclude that for the purpose of fire protection of good quality in all 15 settlements, more than one fire station needs to be built. The problem of area coverage with an optimal number of fire stations is a completely different problem (for example, see [12]).

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