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Tiling by incongruent equilateral triangles without requiring local finiteness, Part II

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Christian Richter, Jahrgang 1969, arbeitet seit 1995 überwiegend an seiner Heimatuniversität in Jena, wo er auch promoviert, habilitiert und kürzlich zum außerplanmäßigen Professor ernannt wurde. Für einige Jahre war er am Institut de Mathématiques de Jussieu, Paris, und an der Universität Magdeburg tätig. Er forscht zu Fragen der diskreten und Konvexgeometrie.

1 Introduction and main theorem

What sets $S \subseteq \mathbb{R}^2$ can be represented as a union of non-overlapping equilateral triangles of mutually different size? The answer will heavily depend on whether the family of triangles is required to be locally finite.

Our question belongs to the large theory of geometric tilings in the Euclidean plane (see [5] for an overview). A *cover* of a set $S \subseteq \mathbb{R}^2$ is a family $\mathcal{T} = \{T_i : i \in I\}$ of sets $T_i \subseteq \mathbb{R}^2$ such that $S \subseteq \bigcup_{i \in I} T_i$. The cover $\mathcal{T}$ is called a *tiling* of S if the sets T_i, now called *tiles*, are non-overlapping and subsets of S; hence $S = \bigcup_{i \in I} T_i$. *Non-overlapping* means that the tiles T_i have mutually disjoint interiors: $\operatorname{int}(T_i) \cap \operatorname{int}(T_j) = \emptyset$ for $i, j \in I$, $i \neq j$. We are interested in the situation where all tiles are equilateral triangles. A tiling by equilateral triangles is called *perfect* if the tiles are of pairwise different size. The notion of perfectness goes back to [1] in the context of tilings by squares.

A tiling or, more generally, a family of subsets of $\mathbb{R}^2$ is called *locally finite* if every bounded subset of $\mathbb{R}^2$ has a nonempty intersection with at most finitely many members of the family.

Kann die euklidische Ebene in paarweise inkongruente gleichseitige Dreiecke zerlegt werden? In der Tat kennt man keine konvexe Menge, die *lokal endlich* mit verschieden großen gleichseitigen Dreiecken gepflastert werden kann. Die Situation ändert sich, wenn auf die lokale Endlichkeit verzichtet wird: Ist eine Punktmenge in gleichseitige Dreiecke zerlegbar, so kann die Zerlegung zu einer Pflasterung mit paarweise inkongruenten gleichseitigen Dreiecken verfeinert werden. Dies erlaubt unter anderem die Charakterisierung aller Polygone und aller abgeschlossenen konvexen Mengen, die so eine Zerlegung zulassen.

Most research on tilings concerns the locally finite case. The following is known on perfect tilings by equilateral triangles. No convex set in $\mathbb{R}^2$ has a perfect tiling by finitely many (and at least two) equilateral triangles (see [11, p. 468] for tilings of equilateral triangles and [2, 12] for the general case). It is an open problem if the whole plane $\mathbb{R}^2$ admits a locally finite perfect tiling by equilateral triangles [5, Exercise 2.4.10], [3, Section C11]. Scherer [9] shows that such a tiling cannot contain a smallest triangle.

Klaaßen [6] drops the condition of local finiteness and presents a perfect tiling by equilateral triangles of the plane $\mathbb{R}^2$ except for one point, which is the only accumulation point of the tiles. Similarly, he finds a tiling of an equilateral triangle except for three points. In [7] we give a general construction of perfect tilings by equilateral triangles, this way obtaining tilings of a complete equilateral triangle, of the whole plane and, more generally, of arbitrary open subsets of $\mathbb{R}^2$. These tilings have uncountably many accumulation points.

The main result of the present paper reads as follows.

Theorem 1. *The following are equivalent for every subset $S \subseteq \mathbb{R}^2$.*

(i) *S admits a perfect tiling by equilateral triangles.*

(ii) *S admits a tiling by equilateral triangles.*

(iii) *There exists a family of non-overlapping equilateral triangles $E_i \subseteq S$, $i \in I$, that covers $S \setminus \operatorname{int}(S)$.*

The proof will be given in the following section. Theorem 1 has two important aspects. The equivalence of (i) and (ii) shows that the existence of a tiling by equilateral triangles implies always the existence of a perfect tiling if we do not require local finiteness. The negative results mentioned above illustrate that the situation is different for locally finite tilings. The equivalent condition (iii) serves as a main tool for finding sets that admit perfect tilings by equilateral triangles. Applications will be given in Section 3, where we characterize in particular all polygons and all closed convex sets having such tilings.

2 Proof of Theorem 1

We use the following notation: $\operatorname{bd}(S)$, $\operatorname{cl}(S)$ and $\operatorname{int}(S)$ denote boundary, closure and interior of a set $S \subseteq \mathbb{R}^2$. $\|\cdot\|$ stands for the Euclidean norm. The sets

$$D(x_0, \varrho) = \{x \in \mathbb{R}^2 : \|x - x_0\| \leq \varrho\}, \quad D^\circ(x_0, \varrho) = \{x \in \mathbb{R}^2 : \|x - x_0\| < \varrho\}$$

and

$$C(x_0, \varrho) = \{x \in \mathbb{R}^2 : \|x - x_0\| = \varrho\}$$

are the closed disc, the open disc and the circle centered at $x_0 \in \mathbb{R}^2$ of radius $\varrho > 0$, respectively.

The proof of Theorem 1 is prepared by four lemmas, two of them from [7]. The first one represents a particular case of the crucial theorem from [7]. It gives a sufficient condition for the existence of perfect tilings, which is based on decompositions into so-called auxiliary pieces. An *auxiliary piece of size $\alpha > 0$* is obtained from a right triangle with edges of lengths α, $\sqrt{3}\alpha$ and 2α by completely removing its edge of length $\sqrt{3}\alpha$ (see Figure 1(a)).

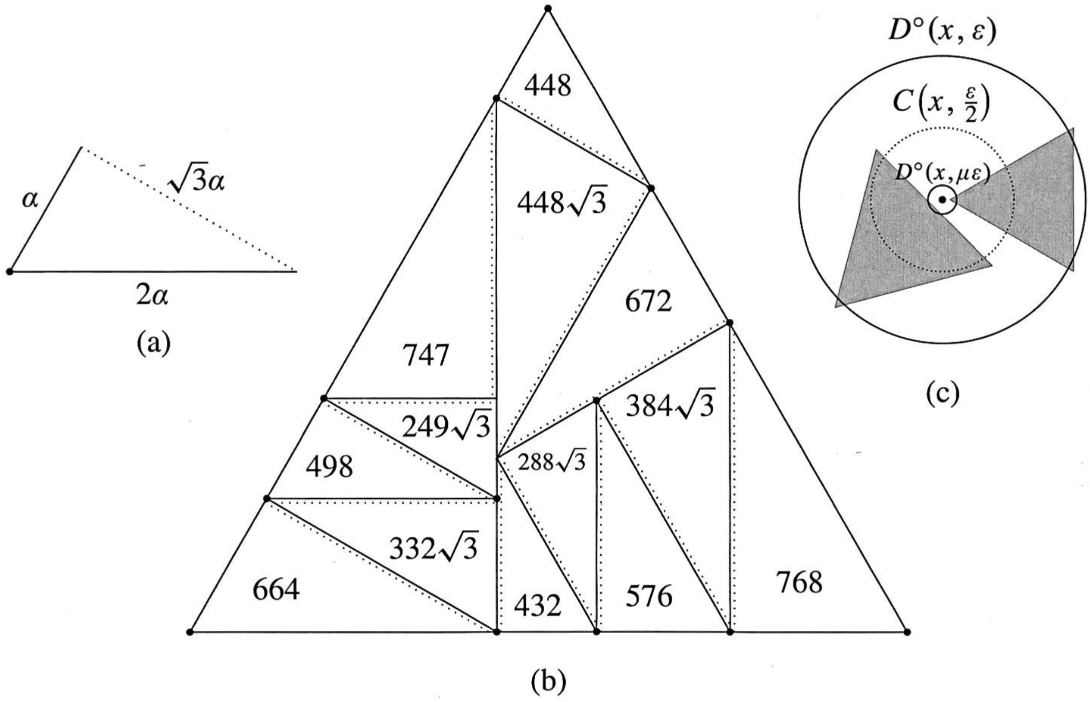

Figure 1 Auxiliary piece of size α (a); proofs of Lemma 4 (b) and Lemma 5 (c)

Lemma 2 (see [7, pp. 158–159]). *Suppose a set $S \subseteq \mathbb{R}^2$ has a representation*

$$S = \bigcup_{A \in \mathcal{A}} A = \bigcup_{A \in \mathcal{A}} \mathrm{cl}(A)$$

where $\mathcal{A}$ is a family of auxiliary pieces with pairwise disjoint interiors. Then S admits a perfect tiling by equilateral triangles.

Lemma 3 ([7, Corollary 2]). *Every open set $G \subseteq \mathbb{R}^2$ admits a perfect tiling by equilateral triangles.*

Lemma 4. *Every equilateral triangle can be decomposed into* 13 *non-overlapping auxiliary pieces.*

Proof. Figure 1(b) illustrates a decomposition of an equilateral triangle of edge length 3104 and displays the sizes of the auxiliary pieces. The removed edge of every auxiliary piece is dotted, the remaining vertex is emphasized, as in Figure 1(a). □

Lemma 5. *There exists a constant $\mu > 0$ with the following property: if no member of a family $\mathcal{E}$ of non-overlapping equilateral triangles is completely contained in the open disc $D^\circ(x, \varepsilon)$, then there are at most six triangles from $\mathcal{E}$ that have a nonempty intersection with $D^\circ(x, \mu\varepsilon)$.*

Proof. If $\mu > 0$ is chosen sufficiently small, then every equilateral triangle that has a point (and in turn a vertex) outside $D^\circ(x, \varepsilon)$ and meets $D^\circ(x, \mu\varepsilon)$ covers at least almost one sixth of $C\left(x, \frac{\varepsilon}{2}\right)$ (see Figure 1(c); for a quantification of μ, see [4, Section 3.1]). Hence not more than six non-overlapping triangles can have that property simultaneously. □

Proof of Theorem 1. The implication (i)$\Rightarrow$(ii) is trivial. For the converse (ii)$\Rightarrow$(i), we use Lemma 4 to decompose every equilateral triangle into auxiliary pieces. Application of Lemma 2 to the resulting family $\mathcal{A}$ of auxiliary pieces proves (i).

The claim (ii)$\Rightarrow$(iii) is obvious. For (iii)$\Rightarrow$(ii), we can suppose that every E_i, $i \in I$, contains a point from $\mathrm{bd}(S)$. It is enough to show that the set $G = S \setminus \bigcup_{i\in I} E_i$ is open, because then Lemma 3 gives a tiling $\mathcal{T}_G$ of G by equilateral triangles and the tiling $\mathcal{T}_G \cup \{E_i : i \in I\}$ of S confirms (ii). To see that G is open, we consider an arbitrary point $x_0 \in G$.

Case 1. *For every $\varepsilon > 0$, there exists $i \in I$ such that $E_i \subseteq D^\circ(x_0, \varepsilon)$.* Since every E_i meets $\mathrm{bd}(S)$, x_0 is an accumulation point of $\mathrm{bd}(S)$ and in turn $x_0 \in \mathrm{bd}(S)$. This gives $G \cap (\mathrm{bd}(S) \cap S) \neq \emptyset$ and contradicts $\mathrm{bd}(S) \cap S = S \setminus \mathrm{int}(S) \subseteq \bigcup_{i\in I} E_i$.

Case 2. *There exists $\varepsilon_0 > 0$ such that $D^\circ(x_0, \varepsilon_0)$ does not cover any of the triangles E_i, $i \in I$.* Now Lemma 5 shows that $D^\circ(x_0, \mu\varepsilon_0)$ meets at most six triangles E_i, $i \in I$, and $D^\circ(x_0, \mu\varepsilon_0) \setminus \bigcup_{i\in I} E_i$ is open. We see that $\left(D^\circ(x_0, \mu\varepsilon_0) \setminus \bigcup_{i\in I} E_i\right) \cap \mathrm{int}(S)$ is an open neighborhood of x_0 in G, because

$$\begin{aligned}\left(D^\circ(x_0, \mu\varepsilon_0) \setminus \bigcup_{i\in I} E_i\right) \cap \mathrm{int}(S) &= D^\circ(x_0, \mu\varepsilon_0) \cap \left(\mathrm{int}(S) \setminus \bigcup_{i\in I} E_i\right)\\ &= D^\circ(x_0, \mu\varepsilon_0) \cap G.\end{aligned}$$

Thus, G is open, and the proof is complete. □

3 Applications

3.1 Polygons

By a *proper convex polygon* we mean the convex hull of finitely many points of $\mathbb{R}^2$ that are not collinear. A *proper polygon* is a union of finitely many proper convex polygons. Proper polygons need not be connected and may have holes. More generally, we call a union of a locally finite family of proper convex polygons a *generalized proper polygon*. These objects may be unbounded, but still are closed.

Given a generalized proper polygon $P \subseteq \mathbb{R}^2$ and a point $x \in \mathbb{R}^2$, the *tangent cone* of P at x is defined as

$$\begin{aligned}\mathrm{Tan}(P; x) = \Big\{ u \in \mathbb{R}^2 \setminus \{(0,0)\} &: \text{there is a sequence } (p_i)_{i=1}^\infty \subseteq P \setminus \{x\} \text{ such}\\ &\text{that } \lim_{i\to\infty} p_i = x \text{ and } \lim_{i\to\infty} \frac{p_i - x}{\|p_i - x\|} = \frac{u}{\|u\|} \Big\} \cup \{(0,0)\}\end{aligned}$$

[8, p. 145]. $\mathrm{Tan}(P; x)$ is a cone with apex $(0,0)$, see Figure 2(a) for an illustration with $x = (0,0)$. The point x is called a *vertex* of P if $\mathrm{Tan}(P; x)$ is neither $\{(0,0)\}$ (then $x \notin P$) nor $\mathbb{R}^2$ (then $x \in \mathrm{int}(P)$) nor a half-plane (then x belongs to the relative interior of an edge of P).

The tangent cone $\mathrm{Tan}(P; v)$ at a vertex v represents in some sense the inner angle of P at v and may be composed of several (but finitely many) angular regions. If P admits a

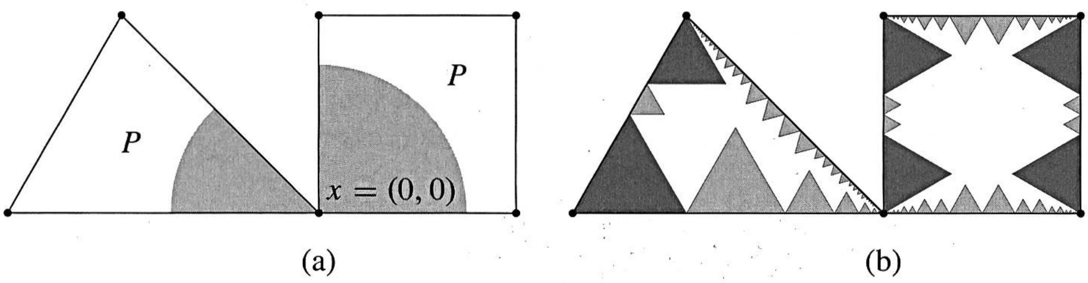

Figure 2 Tangent cone at $x = (0, 0)$ (a); proof of Proposition 6 (b)

tiling by equilateral triangles, then one of these regions must be at least of size $\frac{\pi}{3}$, since v belongs to a triangle of the tiling. This simple condition turns out to be sufficient for the existence of tilings.

Proposition 6. *A generalized proper polygon P admits a perfect tiling by equilateral triangles if and only if, for every vertex v of P, there is an equilateral triangle E such that $v \in E \subseteq P$.*

Proof. The necessity is shown above.

For verifying the sufficiency, we show that P satisfies condition (iii) from Theorem 1. Let V be the set of all vertices of P. The connected components of $\mathrm{bd}(P) \setminus V$ are called the (relatively open) *edges* of P. Note that the set V as well as the family of all edges of P are locally finite. This allows to choose pairwise disjoint equilateral triangles $E_v \subseteq P$, $v \in V$, such that $v \in E_v$ and E_v is disjoint from all edges that do not emanate from v (see the darker triangles in Figure 2(b)). Now $\mathrm{bd}(P) \setminus \bigcup_{v \in V} E_v$ consists of (remainders of) edges of P. For every such segment we pick (possibly infinitely many) equilateral triangles in P that cover the whole segment. This is done such that the triangles covering one segment do not overlap any of the triangles E_v, $v \in V$, nor of the triangles that cover other segments (see the lighter triangles in Figure 2(b)). We have verified condition (iii) from Theorem 1, and the proof is complete. □

Proposition 6 gives an affirmative answer to a question posed in [7, p. 161]. For convex polygons it reads as follows.

Corollary 7. *A proper convex polygon admits a perfect tiling by equilateral triangles if and only if each of its inner angles is not smaller than $\frac{\pi}{3}$.*

3.2 Closed convex sets

Recall that e is an exposed point of a closed convex set $C \subseteq \mathbb{R}^2$ if and only if the singleton $\{e\}$ is the intersection of C with some tangent line of C. $\exp(C)$ denotes the set of all exposed points of C.

Proposition 8. *A closed convex set $C \subseteq \mathbb{R}^2$ admits a perfect tiling by equilateral triangles if and only if $\exp(C)$ is at most countable and, for every $e \in \exp(C)$, there is an equilateral triangle E such that $e \in E \subseteq C$.*

Proof. Necessity. If C has a tiling $\mathcal{T}$ by triangles, every exposed point e of C must be a vertex of a tile $E \in \mathcal{T}$. This shows in particular that $\exp(C)$ is at most countable, because $\mathcal{T}$ cannot be uncountable. To see this, we pick a point with rational coordinates in the interior of every triangle of $\mathcal{T}$, this way obtaining an injective map of $\mathcal{T}$ into the countable set $\mathbb{Q}^2$.

Sufficiency. Step 1*: preparation.* By a *side* of C we mean an intersection of C with a tangent line that contains more than one point (i.e., an *exposed face* of dimension one in the terminology of [10, p. 63]).

C has at most countably many sides. Indeed, each bounded convex set $C \cap D((0,0), n)$, $n \in \mathbb{N}$, has at most countably many sides, because $\mathrm{bd}(C \cap D((0,0), n))$ has finite length. Since every side of C is (at least part of) a side of some $C \cap D((0,0), n)$, there are at most countably many sides of C.

If e is an endpoint of a side S of C, e can be an exposed point of C or the straight line spanned by S is the only tangent of C through e. In either case we know that there exists an equilateral triangle E such that $e \in E \subseteq C$.

C can have up to two unbounded sides. To simplify further discussion, we split every unbounded side into segments of length one. We call the relative interiors of these segments as well as those of all bounded sides of C *b-sides* of C.

We have reached the following situation: $\mathrm{bd}(C)$ *is represented as a disjoint union*

$$\mathrm{bd}(C) = V \cup \bigcup_{B \in \mathcal{B}} B.$$

Here $\mathcal{B}$ is a family of bounded and relatively open line segments, called b-sides, and V is an at most countable set, consisting of all exposed points of C and all endpoints of b-sides. For every $v \in V$, there exists an equilateral triangle E such that $v \in E \subseteq C$.

Step 2*: construction.* We suppose that $V = \{v_1, v_2, \ldots\}$ is infinite. (Otherwise C is a convex polygon, since C is bounded and $\exp(C)$ is finite, and Corollary 7 applies.) Next we construct *non-overlapping equilateral triangles $E_i \subseteq C$, $i = 1, 2, \ldots$, and trapezoids $T_B \subseteq C$, $B \in \mathcal{B}$, such that*

(A) *for all $i = 1, 2, \ldots$, $E_i \cap V = \{v_i\}$ and if E_i meets some $B \in \mathcal{B}$, then v_i is an endpoint of B,*

(B) *for all $B \in \mathcal{B}$, $T_B \cap \mathrm{bd}(C) = \mathrm{cl}(\tilde{B})$ where $\tilde{B} = B \setminus \bigcup_{i=1}^{\infty} E_i$ is a relatively open segment such that $\mathrm{cl}(\tilde{B})$ is an edge of T_B.*

We proceed by induction on i. In the first step we pick a sufficiently small triangle $E_1 \subseteq C$ that satisfies (A). Now suppose that $E_1, \ldots, E_{i-1}$ and at most finitely many trapezoids T_B are constructed in the first $i-1$ steps of the induction and let the polygon P_{i-1} be the union of these triangles and trapezoids. In the ith step we fix a small triangle $E_i \subseteq C \setminus P_{i-1}$ that satisfies (A). v_i might be an endpoint of a b-side $B \in \mathcal{B}$ whose other endpoint v_j is already covered by some triangle, i.e., $j < i$. This happens for at most two b-sides. Figure 3 illustrates the situation with $i = 5$, $j = 3$. Then the remainder $\tilde{B} = B \setminus (E_i \cup E_j) = B \setminus (P_{i-1} \cup E_i)$ of B constitutes a (relatively open) edge of a trapezoid $T_B \subseteq \mathrm{int}(C) \cup \mathrm{cl}(\tilde{B})$

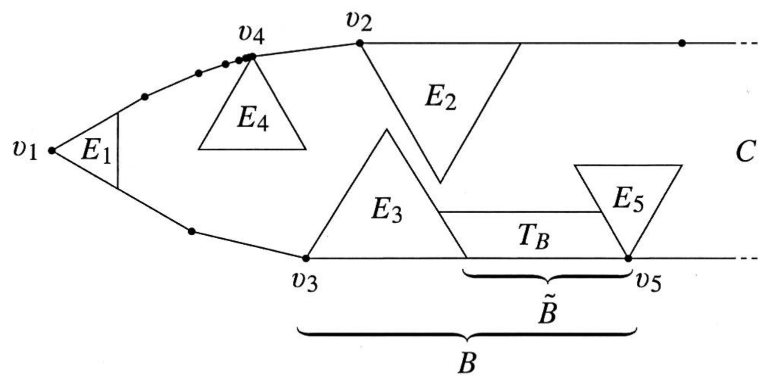

Figure 3 Proof of Proposition 8

whose edge parallel to $\tilde{B}$ connects E_i with E_j such that T_B does not overlap P_{i-1}. This finishes step 2.

Step 3: conclusion. The proof is complete once we have shown that C satisfies condition (iii) from Theorem 1. The triangles E_i, $i = 1, 2, \ldots$, cover all points of V. For every segment $\tilde{B}$, we define a sequence of non-overlapping equilateral triangles $E_i^B \subseteq T_B$, $i = 1, 2, \ldots$, such that $\tilde{B} \subseteq \bigcup_{i=1}^{\infty} E_i^B$. This can be done as in Figure 2(b). Now the triangles of $\{E_i : i = 1, 2, \ldots\} \cup \left\{E_i^B : B \in \mathcal{B}, i = 1, 2, \ldots\right\}$ are subsets of C, do not overlap and cover $\mathrm{bd}(C)$. This completes the proof. □

3.3 Sets containing few of their boundary points

The following generalizes Lemma 3 (see [7, Corollary 2]).

Proposition 9. *If $S \subseteq \mathbb{R}^2$ is a set such that $S \setminus \mathrm{int}(S)$ is at most countable and, for every $x \in S \setminus \mathrm{int}(S)$, there is an equilateral triangle E such that $x \in E \subseteq S$, then S admits a perfect tiling by equilateral triangles.*

Proof. Let $S \setminus \mathrm{int}(S) = S \cap \mathrm{bd}(S) = \{b_i : i = 1, 2, \ldots\}$. (In fact, $S \setminus \mathrm{int}(S)$ might be finite. This would simplify the situation.) We prove condition (iii) from Theorem 1 by induction on i. For $i = 1$ we pick some equilateral triangle E_1 such that $b_1 \in E_1 \subseteq S$. For the ith step, suppose that we have already chosen disjoint triangles $E_j \subseteq S$, $1 \leq j \leq k$, such that $\{b_1, \ldots, b_{i-1}\} \subseteq E_1 \cup \cdots \cup E_k$. If b_i is contained in $E_1 \cup \cdots \cup E_k$, then we proceed with step $i + 1$ immediately. Otherwise we pick a small equilateral triangle E_{k+1} disjoint from $E_1 \cup \cdots \cup E_k$ such that $b_i \in E_{k+1} \subseteq S$ before we continue with step $i + 1$. This way we obtain condition (iii) from Theorem 1, which completes the proof. □

References

[1] R.L. Brooks, C.A.B. Smith, A.H. Stone, W.T. Tutte: *The dissection of rectangles into squares*, Duke Math. J. **7** (1940), 312–340.

[2] E. Buchman: *The impossibility of tiling a convex region with unequal equilateral triangles*, Amer. Math. Monthly **88** (1981), 748–753.

[3] H.T. Croft, K.J. Falconer, R.K. Guy: *Unsolved problems in geometry*, Problem Books in Mathematics. Unsolved Problems in Intuitive Mathematics, II. Springer-Verlag, New York (1991).

[4] M. Förster: *Perfekte Zerlegungen in gleichseitige Dreiecke*, Thesis, Institute of Mathematics, Friedrich Schiller University of Jena, Germany (2016).

[5] B. Grünbaum, G.C. Shephard: *Tilings and patterns*, W.H. Freeman and Company, New York, 1987.

[6] B. Klaaßen: *Infinite perfekte Dreieckszerlegungen auch für gleichseitige Dreiecke*, Elem. Math. **50** (1995), 116–121.

[7] C. Richter: *Tiling by incongruent equilateral triangles without requiring local finiteness*, Elem. Math. **67** (2012), 157–163.

[8] R.T. Rockafellar: *Clarke's tangent cones and the boundaries of closed sets in* $\mathbb{R}^n$, Nonlinear Anal. **3** (1979), 145–154.

[9] K. Scherer: *The impossibility of a tesselation of the plane into equilateral triangles whose sidelengths are mutually different, one of them being minimal*, Elem. Math. **38** (1983), 1–4.

[10] R. Schneider: Convex bodies: the Brunn–Minkowski theory. Encyclopedia of mathematics and its applications, vol. 44, Cambridge University Press, Cambridge (1993).

[11] W.T. Tutte: *The dissection of equilateral triangles into equilateral triangles*, Proc. Cambridge Philos. Soc. **44** (1948), 463–482.

[12] Z. Tuza: *Dissections into equilateral triangles*, Elem. Math. **46** (1991), 153–158.

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