THE NUMBERS OF ZEROS AND OF CHANGES OF SIGN IN A SYMMETRIC RANDOM WALK

Autor(en): **Feller, William**

Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **3 (1957)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **03.06.2024**

Persistenter Link: https://doi.org/10.5169/seals-33746

THE NUMBERS OF ZEROS AND OF CHANGES OF SIGN IN A SYMMETRIC RANDOM WALK *

BY

William FELLER, Princeton University

1. INTRODUCTION.

After a very long period of oblivion, the theory of the symmetric random walk once more attracts widespread attention. Curious and totally unexpected fluctuation phenomena have been discovered and described in the arc sine law and other equally preposterous theorems [1]. As E. Sparre Andersen has shown [2], these laws apply to an exceedingly large class of chance processes and they have completely revolutionized our notions concerning chance fluctuations when cumulative effects are involved.

Most of these newly discovered theorems are related to the consecutive returns to the origin. The present paper has the modest purpose of deriving *explicit formulas for the probability distributions of the number of returns to the origin*, *the number of changes of sign*, etc. during the first n steps of a symmetric random walk in one dimension. The limiting form of these distributions as $n \to \infty$ are known, and therefore no special importance can be ascribed to the knowledge of the explicit formulas. However, they are pleasing and surprisingly simple.

---

* This research was done while the author was on sabbatical leave of absence with the support of the United States Air Force through the Air Force Office of Scientific Research and Development Command under Contract No. AF18 (603)-24 with Princeton University.

[1] See, for example, *An Introduction to Probability Theory and its Applications* by William FELLER, New York, 1950, Chapter 12. A more exhaustive treatment will be contained in Chapter III of the second edition of this book.

[2] On the fluctuations of sums of random variables I and II, *Mathematica Scandinavica*, vol. 1 (1953), pp. 263-285 and vol. 2 (1954), pp. 195-223.

Furthermore, the derivation is of a quite elementary nature and therefore of some independent interest. In fact, we shall start from the simple combinatorial formula (2.4) and from it derive all results by a direct procedure without presupposing any knowledge concerning random walks and without using any analytical tools.

2. Preparations.

Let X_1, X_2, ... denote an infinite sequence of mutually independent random variables each assuming the values ± 1 with probability $\frac{1}{2}$. Put

$$S_0 = 1, \qquad S_n = X_1 + X_2 + \dots + X_n \qquad (n \geq 1) \tag{2.1}$$

Then S_n is to be interpreted as the coordinate, at time n (or after n steps), in a one-dimensional symmetric random walk starting from the origin. *A return to the origin occurs at time n if* $S_n = 0$. Obviously n must be even. For the probability of such a return we write

$$u_n = P\{S_n = 0\}, \qquad u_0 = 1 \tag{2.2}$$

Clearly

$$u_{2n} = \frac{1}{2^{2n}}\binom{2n}{n}, \qquad u_{2n-1} = 0\,. \tag{2.3}$$

All our considerations will depend on the following simple and well known Lemma:

$$\sum_{r=0}^{n} u_{2r}\, u_{2n-2r} = 1\,. \tag{2.4}$$

Proof. We introduce the generating function

$$U(s) = \sum_{n=0}^{\infty} u_{2n}\, s^{2n} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} s^{2n} = (1 - s^2)^{-\frac{1}{2}}\,. \tag{2.5}$$

It is then clear that the left side on (2.4) equals the coefficient of s^{2n} in $U^2(s) = (1 - s^2)^{-1}$, and thus (2.4) is true.

3. The probability of no zeros.

Let p_n *be the probability that the first n steps do not lead to a return to the origin, that is,*

$$p_n = \mathrm{P}\{ S_1 \neq 0 , \quad S_2 \neq 0, \ldots, S_n \neq 0 \}, \qquad p_0 = 1 . \tag{3.1}$$

Theorem 1. *We have*

$$p_{2n} = u_{2n} \qquad n = 0, 1, 2, \ldots \tag{3.2}$$

Proof. Denote by A_r the event that among the partial sums $S_0, S_1, \ldots, S_{2n}$ the *last zero* has index $2r$:

$$A_r = \{ S_{2r} = 0, \quad S_{2r+1} \neq 0, \quad S_{2r+2} \neq 0, \ldots, \quad S_{2n} \neq 0 \} \tag{3.3}$$

$$= \{ S_{2r} = 0 \} \cap \{ S_{2r+1} - S_{2r} \neq 0 , \quad S_{2r+2} - S_{2r} \neq 0, \ldots, \quad S_{2n} - S_{2r} \neq 0 \}$$

The two events on the extreme right side are independent and have, respectively, probabilities u_{2r} and p_{2n-2r}. The union of the events A_r covers the sample space of the sequences $S_0, \ldots, S_n$, and these events are mutually exclusive. Therefore

$$1 = \sum_{r=0}^{n} \mathrm{P}\{ A_r \} = \sum_{r=0}^{n} u_{2r}\, p_{2n-2r} \tag{3.4}$$

and a comparison of (3.4) with (3.1) proves the theorem.

The last theorem is fully equivalent to the following corollary which is well known.

Corollary. *Let* f_{2n} *be the probability that the first return to the origin takes place at the* $2n$*-th step, that is,*

$$f_{2n} = \mathrm{P}\{ S_1 \neq 0, \quad S_2 \neq 0, \ldots, S_{2n-1} \neq 0, S_{2n} = 0 \} . \tag{3.5}$$

Then

$$f_{2n} = u_{2n} - u_{2n-2} . \tag{3.6}$$

Proof. From (3.1) and (3.5) it is obvious that $f_{2n} = u_{2n} - u_{2n-2}$.

4. The number of zeros.

Theorem 2. *For $n \geq 1$ let $z_{k,n}$ be the probability that exactly k among the n partial sums $S_1, \ldots, S_n$ vanish. For $n = 0$ put*

$$z_{0,0} = 1, \qquad z_{1,0} = z_{2,0} = \ldots = 0. \tag{4.1}$$

Then

$$z_{k,2n} = \frac{2^k}{2^{2n}} \binom{2n-k}{n}. \tag{4.2}$$

Proof. By definition

$$z_{0,2n} = p_{2n} = u_{2n}, \qquad (n \geq 0). \tag{4.3}$$

To evaluate $z_{1,2n}$ denote by B_r the event that among the partial sums $S_1, \ldots, S_{2n}$ exactly one vanishes and its index equals $2r$. Then for $r < n$

$$\begin{aligned} B_r &= \{ S_1 \neq 0, \ldots, S_{2r-1} \neq 0, S_{2r} = 0, S_{2r+1} \neq 0, \ldots, S_{2n} \neq 0 \} \\ &\equiv \{ S_1 \neq 0, \ldots, S_{2r-1} \neq 0, S_{2r} = 0 \} \cap \{ S_{2r+1} - S_{2r} \neq 0, \ldots, S_{2n} - S_{2r} \neq 0 \}. \end{aligned}$$

Since the two events on the right are stochastically independent and the B_r are mutually exclusive we conclude that

$$z_{1,2n} = \sum_{r=1}^{n} P\{ B_r \} = \sum_{r=1}^{n} f_{2r} z_{0,2n-2r}. \tag{4.5}$$

Now by Theorem 1 the last event on the right in (4.4) has the same probability as the event $\{ S_{2n} - S_{2r} = 0 \}$ and hence we have for $r \leq n$

$$P\{ B_r \} = P\{ S_1 \neq 0, \ldots, S_{2r-1} \neq 0, S_{2r} = 0, S_{2n} = 0 \}. \tag{4.6}$$

The events appearing on the right side are mutually exclusive and their union is the event $\{ S_{2n} = 0 \}$; hence

$$\sum_{r=1}^{n} P\{ B_r \} = P\{ S_{2n} = 0 \} = u_{2n}. \tag{4.7}$$

Comparing (4.5) and (4.7) we see that

$$z_{1,2n} = u_{2n} = z_{0,2n} \qquad \textit{for } n \geq 1 . \tag{4.8}$$

In like manner we can calculate $z_{2,\,2n}, z_{3,\,2n}, \ldots$ from the recursion formula

$$z_{k,2n} = \sum_{r=1}^{n-1} f_{2r}\, z_{k-1,2n-2r}\,, \qquad k \geq 2, \quad n \geq 1 . \tag{4.9}$$

which is proved exactly as (4.5). For $k \geq 2$ the right side differs from the right side in (4.5) only in that the term $r = n$ is absent, and therefore

$$z_{k,2n} = z_{1,2n} - f_{2n} = 2 z_{1,2n} - z_{0,2n-2}\,, \qquad n \geq 1 . \tag{4.10}$$

From the last two relations we see directly by induction that *for $k \geq 2$ and $n \geq 1$ we have the recursion formula*

$$z_{k,2n} = 2 z_{k-1,2n} - z_{k-2,2n-2} \tag{4.11}$$

If we write $z_{k,\,2n} = 2^{k-2n}\, a_{k,\,2n}$ then (4.11) reduces to

$$a_{k-1,2n} = a_{k,2n} + a_{k-2,2n-2} \tag{4.12}$$

which is the well-known addition relation for binomial coefficients, and thus (4.2) holds.

This theorem has the following surprising

Corollary. *For each $n \geq 1$ we have*

$$z_{0,2n} = z_{1,2n} > z_{2,2n} > z_{3,2n} > \cdots > z_{n,2n} \tag{4.13}$$

Thus, independently of the number n of steps, the *most probable number of zeros is* 0, and the smaller the number, the more probable it is.

5. The number of changes of sign.

We say that in the sequence $S_1, \ldots, S_{2n}$ *a change of sign occurs at the place* j if S_{j-1} and S_{j+1} are of opposite signs. This requires that $S_j = 0$, and so j must be even. Given the first $2n$ terms of the sequence we can speak of changes of sign only at the places $j \leq 2n - 2$.

THEOREM 3. *Let $c_{r,2n}$ denote the probability that there exist exactly r indices j such that*

$$S_{j-1} S_{j+1} < 0 , \qquad 1 \leq j \leq 2n - 1 . \tag{5.1}$$

Then

$$c_{r,2n} = \frac{1}{2^{2n-2}} \binom{2n-1}{n-1-r} . \tag{5.2}$$

Proof. Let us say that two sequences $S_1, \ldots, S_m$ and $S'_1, \ldots, S'_m$ are *similar* if $|S_j| = |S'_j|$ for $j = 1, 2, \ldots, m$. Obviously $-S_1, -S_2, \ldots, -S_{2n}$ represents the only sequence similar to $S_1, \ldots, S_{2n}$ and such that changes of sign occur at the same places. On the other hand, if exactly k among the terms $S_1, \ldots, S_{2n-2}$ vanish, there exist exactly 2^{k+1} sequences similar to the sequence $S_1, \ldots, S_{2n}$. Out of k places we may choose r places in $\binom{k}{r}$ different ways, and it is therefore seen that

$$\begin{aligned} c_{r,2n} &= 2 \sum_{k=r}^{n-1} \binom{k}{r} 2^{-(k+1)} z_{k,2n-2} \\ &= \frac{1}{2^{2n-2}} \sum_{k=r}^{n-1} \binom{k}{r} \binom{2n-2-k}{n-1} . \end{aligned} \tag{5.3}$$

A well-known formula for binomial coefficients [3] which can be proved by induction now shows that (5.2) is true.

In (5.2) we recognize the binomial distribution and we have the obvious.

COROLLARY:

$$c_{0,2n} > c_{1,2n} > c_{2,2n} > \ldots > c_{n-1,2n} . \tag{5.4}$$

6. THE EXPECTATIONS.

THEOREM 4. *Let Z_{2n} and C_{2n} denote, respectively, the number of zeros and the number of changes of sign among the terms $S_1, \ldots,$*

[3] See, for example, formula (9.11) of Chapter 2 of the book quoted above.

S_{2n-1}. *For the expectations of these random variables we have*

$$2\,E\,(C_{2n}) = E\,(Z_{2n}) = 2nu_{2n} - 1 \tag{6.1}$$

and thus

$$2\,E\,(C_{2n}) = E\,(Z_{2n}) \sim \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (2n)^{\frac{1}{2}} \quad \textit{as } n \longrightarrow \infty\,. \tag{6.2}$$

(These formulas shows that the density of the zeros and of changes of sign decreses at a fast rate.)

Proof. Define new random variables by $Y_j = 1$ if $X_j = 0$, and $Y_j = 0$ if $X_j \neq 0$. Then

$$2\,E\,(C_{2n}) = E\,(Z_{2n}) = \sum_{j=1}^{2n-1} E\,(Y_j) = \sum_{r=1}^{n-1} u_{2r}\,. \tag{6.3}$$

and (6.1) follows by induction.

7. Later returns to the origin.

As a further application of the present elementary approach let us prove an important formula half of which has been proved by rather involved analytical methods [4].

Theorem 5. *Let $f_{k,2n}$ denote the probability that the k-th return to the origin takes place at the $2n$-th step (that is, $f_{k,2n}$ is the probability that $S_{2n} = 0$ and exactly $r - 1$ among the S_j with $1 \leq j < 2n$ vanish). Then*

$$f_{k,2n} = z_{k,2n} - z_{k+1,2n} = \frac{2^k}{2^{2n}} \binom{2n-k}{n} \frac{k}{2n-k}\,. \tag{7.1}$$

Proof. It is clear that $f_{1,2n} = f_{2n}$ and that the $f_{k,2n}$ satisfy the recurrence relation (4.9) and hence also (4.11). If we define $f_{0,2n} = 0$ for n 1 and $f_{0,0} = 1$, then (7.1) is true for $k = 0,1$ and therefore for all $k \geq 0$.

Reçu le 17 mai 1957.

[4] See, for example, *ibid.*, formula (6.15) of Chapter 12.