1. Introduction.

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **4 (1958)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

ON CERTAIN ARITHMETICAL FUNCTIONS RELATED TO A NON-LINEAR PARTIAL DIFFERENTIAL EQUATION [1)]

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(Reçu le 21 mars 1957)

1. INTRODUCTION.

In a recent paper, VAN DER POL [1] has made an extensive study of the elliptic modular functions defined by:

$$(1) \qquad \alpha_{2k-1}(t) = \frac{\sum\limits_{m=-\infty}^{\infty}{}' \sum\limits_{n=-\infty}^{\infty} (m + n\tau)^{-2k}}{\sum\limits_{m=-\infty}^{\infty}{}' m^{-2k}}, \quad k = 2, 3, 4, 5, \ldots$$

where $-t = 2\pi i\tau$, $Im\tau > 0$; m, n range over all integral values and $(m, n) \neq (0, 0)$. HURWITZ [2] and VAN DER POL [1] have shown by different methods that these functions have series representations of the form

$$(2) \quad \alpha_{2k-1}(t) = 1 + \frac{4(-1)^k_k}{B_k} \sum_{n=1}^{\infty} \frac{n^{2k-1} e^{-nt}}{1 - e^{-nt}} = 1 + \frac{4k(-1)^k}{B_k} \sum_{n=1}^{\infty} e^{-nt} \sigma_{2k-1}(n),$$

where $\sigma_{2k-1}(n)$ is the sum of the $(2k-1)$-*st* powers of the integral divisors of n, and B_k are the Bernoulli numbers. These functions are closely related to the coefficients in the series development of the Weierstrass function $\wp(u)$, and may be found tabulated in VAN DER POL's paper [1] as well as in a paper by RAMANUJAN [3] who uses a different notation.

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1) This paper was prepared while the author held a Faculty Research Fellowship during the summer of 1956, granted by the University of Nebraska Research Council.

The integers (m, n) in (1) are unrestricted as to parity, and it is of some interest to consider the three sequences of functions for which the corresponding defining double sum is restricted by the conditions that

$$(\mathrm{m}, \mathrm{n}) \equiv (0,1),\ (1,0) \text{ or } (1,1), \bmod 2.$$

These functions are defined respectively, for $k > 1$, by

$$(4) \quad \Psi_{2k-1}(t) = C_k \sum_{(\mu)} \sum_{(\nu)} (\mu + \nu\tau)^{-2k},$$

$$\left(\begin{array}{l} \mu = 0 \pm 2,\ \pm 4, \ldots) \\ \nu = \pm 1,\ + 3,\ \pm 5, \ldots \end{array}\right)$$

$$(5) \quad X_{2k-1}(t) = C_k \sum_{(\mu)} \sum_{(\nu)} (\nu + \mu\tau)^{-2k},$$

$$(6) \quad \Phi_{2k-1}(t) = C_k \sum_{(\rho)} \sum_{(\sigma)} (\rho + \sigma\tau)^{-2k}, \qquad (\rho, \sigma = \pm 1,\ \pm 3,\ \pm 5, \ldots).$$

where,

$$(7) \quad C_k \sum_{(\mu)}{}' \mu^{-2k} = V_k, \quad V_k = \frac{(-1)^k B_k}{4k}; \quad C_k = \frac{(-1)^k (2k-1)!}{2^{2k+1} \pi^{2k}}.$$

Written in arithmetical form these functions will be shown to have the form:

$$(4)_1 \quad \Psi_{2k-1}(t) = \sum_{n=1}^{\infty} \frac{n^{2k-1} e^{-nt/2}}{1 - e^{-nt}} = \sum_{n=1}^{\infty} e^{-nt/2} \beta_{2k-1}(n),$$

$$(5)_1 \quad X_{2k-1}(t) = U_k + \sum_{n=1}^{\infty} \frac{(-1)^n n^{2k-1} e^{-nt}}{1 - e^{-nt}} = U_k + \sum_{n=1}^{\infty} e^{-nt} \zeta_{2k-1}(n),$$

$$(6)_1 \quad \Phi_{2k-1}(t) = \sum_{n=1}^{\infty} \frac{(-1)^k n^{2k-1} e^{-nt/2}}{1 - e^{-nt}} = e^{-nt/2} \xi_{2k-1}(n),$$

where,

(8) $\beta_{2k-1}(n)$ = sum of the $(2k-1)$ *st* powers of the integral divisors of n whose conjugates are odd.

(9) $\zeta_{2k-1}(n)$ = (sum of the $(2k-1)$ *st* powers of the even divisors of n) — (sum of the $(2k-1)$ — *st* powers of the odd divisors of n).

(10) $\xi_{2k-1}(n)$ = (sum of the $(2k-1)$ *st* powers of the even divisors of n whose conjugates are odd) — (sum of the $(2k-1)$ — *st* powers of the odd divisors of n whose conjugates are odd);

$$(11) \qquad U_k = (2^{2k} - 1) V_k = (-1)^k (2^{2k} - 1) \frac{B_k}{4k} .$$

As is well known, the double series occurring in (1), (4), (5), (6) are absolutely convergent for $k > 1$; for $k = 1$, the convergence is conditional. However, as has been shown by HURWITZ [2] in the case of (1), if the summation is first carried out with respect to m and then with respect to n, the resulting sum agrees with (2) with $k = 1$. For this case $(k = 1)$ similar conditions hold for (4), (5) and (6). These matters are of relevance in studying certain modular transformations of these functions to be discussed later.

2. UMBRAL RELATIONS.

The functions defined in what precedes arise in a natural manner as a consequence of the well-known fact that the Jacobi theta functions are solutions of the partial differential equation

$$(12) \qquad \frac{\partial^2 z}{\partial s^2} = 2 \frac{\partial z}{\partial t}, \quad z = \theta_r(\nu, \tau), \quad (r = 1, 2, 3, 4),$$

with $s = 2\pi\nu$ and $-t = 2\pi i\tau$, and, what appears to be less well-known, that the functions $u = ln\theta_r(\nu, \tau)$ satisfy the non-linear equation:

$$(13) \qquad \frac{\partial^2 u}{\partial s^2} = 2 \frac{\partial u}{\partial t} - \left(\frac{\partial u}{\partial s}\right)^2 .$$

Here, the notation for the theta function is that used in TANNERY-MOLK's treatise [4].

The arithmetical consequence of (13) can best be obtained through the use of the infinite product representation of $\theta_r(\nu, \tau)$. It is found that the calculations needed are greatly facilitated and the results obtained very simply expressed in a symbolic form through an application of the umbral calculus of BLISSARD and LUCAS [5]. It is not feasible to give details for all cases and we merely indicate briefly the nature of the calculations for the case $r = 4$. Thus, since,