3. Recurrences.

powers of Ψ are lowered into subscripts; thus $\Psi^{(1)}$ would then be written $\Psi_1(t)$.

If (18) and (19) are substituted in (13), there results the following umbral identity:

$$\Psi(1 - \cos \Psi s) + 2 \frac{\partial}{\partial t}\left(\frac{1 - \cos \Psi s}{\Psi}\right) = 2 \sin \Psi s * \sin \Psi s \,, \tag{20}$$

where the asterisk (*) indicates umbral multiplication.

For the cases $r = 2, 3$, rather extensive calculations show that umbral identities of the same form exist. We may therefore state the following result which is implied by the non-linear equation (13).

Theorem 1: "Let Ψ, X, Φ be respectively the umbrae of the sequences $\{\Psi_{2k-1}(t)\}$, $\{X_{2k-1}(t)\}$, and $\{\Phi_{2k-1}(t)\}$. If γ is one of these umbrae, then the following umbral identity holds:

$$\gamma(1 - \cos \gamma s) + 2 \frac{\partial}{\partial t}\left(\frac{1 - \cos \gamma s}{\gamma}\right) = 2 \sin \gamma s * \sin \gamma s \,. \tag{21}$$

3. Recurrences.

It is clear that (20) implies a recurrence relation for the functions $\Psi_j(t)$, and indeed, Theorem 1 yields the following.

Theorem 2: "Let $\gamma_j(t)$ be $\Psi_j(t)$, $X_j(t)$ or $\Phi_j(t)$; then the following recurrence holds:

$$\frac{d}{dt}\gamma_{2n-1}(t) + \frac{1}{2}\gamma_{2n+1}(t) = \sum_{k=0}^{n-1}\binom{2n}{2k+1}\gamma_{2k+1}(t)\,\gamma_{2n-2k-1}(t) \,, \tag{22}$$

and hence $\gamma_{2n+1}(t)$ is a polynomial in $\gamma_1(t)$ and its derivatives up to order n."

This result, in turn, implies the following

Theorem 3: "Let $\rho_{2k-1}(n)$ be either of the arithmetical func tions $\beta_{2k-1}(n)$ or $\xi_{2k-1}(n)$ defined by (8) and (10) respectively; then $\rho_{2k-1}(n)$ satisfies a recurrence relation of the form:

$$\rho_{2k+1}(n) - n\,\rho_{2k-1}(n) = 2\sum_{s=0}^{k-1}\sum_{j=1}^{n-1}\binom{2k}{2s+1}\rho_{2s+1}(j)\,\rho_{2k-2s-1}(n-j), \tag{23}$$

for all n and $k \geq 1$. Moreover, the arithmetical function $\zeta_{2k-1}(n)$ defined by (9) satisfies the recurrence

$$\zeta_{2k+1}(n) - 2n\,\zeta_{2k-1}(n) = 2\sum_{s=0}^{k-1}\binom{2k}{2s+1}\Big\{U_{k-s}\,\zeta_{2s+1}(n) + U_{s+1}\,\zeta_{2k-2s-1}(n) + \sum_{j=1}^{n-1}\zeta_{2s+1}(j)\,\zeta_{2k-2s-1}(n-j)\Big\} \tag{24}$$

where U_k is defined by (11) and $n, k \geq 1$."

Incidentally, the comparison of coefficients which yields (24) also gives:

$$U_{n+1} = 2\sum_{k=0}^{n-1}\binom{2n}{2k+1}U_{k+1}\,U_{n-k}\,, \quad n \geq 1\,, \tag{25}$$

which is equivalent to a result given by NIELSEN [7].

Finally, the case $r = 1$, has been discussed by VAN DER POL [1] who finds an expression analogous to (22) as follows:

$$\frac{d}{dt}h_{2n-1}(t) + \frac{2n+3}{4n+2}h_{2n+1}(t) = \sum_{k=0}^{n-1}\binom{2n}{2k+1}h_{2k+1}(t)\,h_{2n-2k-1}(t)\,, \quad n \geq 1\,, \tag{26}$$

where,

$$h_{2n-1}(t) = \frac{(-1)^n B_n}{4n}\alpha_{2n-1}(t)\,, \tag{27}$$

$\alpha_{2n-1}(t)$ being defined by (2).

We find the analogue of (24) for this case to be:

$$\frac{2k+3}{2k+1}\sigma_{2k+1}(n) - 2n\,\sigma_{2k-1}(n) = 2\sum_{s=0}^{k-1}\binom{2k}{2s+1}\Big\{V_{k-s}\,\sigma_{2s+1}(n) + V_{s+1}\,\sigma_{2k-2s-1}(n) + \sum_{j=1}^{n-1}\sigma_{2s+1}(j)\,\sigma_{2k-2s-1}(n-j)\Big\}\,, \quad n \geq 1\,. \tag{28}$$

Corresponding to (25), we find

$$V_{n+1} = \frac{4n+2}{2n+3}\sum_{k=0}^{n-1}\binom{2n}{2k+1}V_{k+1}\,V_{n-k}\,, \tag{29}$$

which is equivalent to a known recurrence for the BERNOULLI numbers [8].

4. THE FUNCTIONS $\Psi_{2k-1}(t)$, $X_{2k-1}(t)$, $\Phi_{2k-1}(t)$ AS DOUBLE SUMS.

The results which are stated as (4), (5), (6) follow readily from (1) and (2) which are known to be equivalent (see [1], [2]). It is to be observed first that a comparison of (4) and (5) with (1) taking into account (27) gives the relations:

$$\Psi_{2k-1}(t) = h_{2k-1}(t/2) - h_{2k-1}(t) = V_k\left(\alpha_{2k-1}(t/2) - \alpha_{2k-1}(t)\right), \tag{30}$$

$$X_{2k-1}(t) = 2^{2k} h_{2k-1}(2t) - h_{2k-1}(t) = V_k\left(2^{2k}\alpha_{2k-1}(2t) - \alpha_{2k-1}(t)\right). \tag{31}$$

From (4) and (6) we also have,

$$\Phi_{2k-1}(t) = 2^{2k}\Psi_{2k-1}(2t) - \Psi_{2k-1}(t). \tag{32}$$

By (30), we may write

$$\Phi_{2k-1}(t) = -V_k\left(\alpha_{2k-1}(t/2) - (2^{2k}+1)\alpha_{2k-1}(t) + 2^{2k}\alpha_{2k-1}(2t)\right). \tag{33}$$

Thus, our functions (4), (5), (6) are expressed in terms of $\alpha_{2k-1}(u)$. These relations in conjunction with (1) and (2) identify them with $(4)_1$, $(5)_1$, and $(6)_1$ respectively.

It is of interest to note that (31) with $k = 2$ permits, with the aid of a result of VAN DER POL [1], the deduction of Jacobi's famous theorem on the number of representations $r_8(n)$ of the integer n as the sum of eight squares. Thus,

$$240\, X_3(t) = 16\,\alpha_3(2t) - \alpha_3(t) = 15\,\theta_0^8(0, q) \tag{34}$$

where $q = \exp(-t)$. Hence,

$$\theta_0^8(0, q) = 16\, X_3(t) = 1 + 16\sum_{n=1}^{\infty} q^n \zeta_3(n),$$

and

$$\theta_3^8(0, q) = 1 + 16\sum_{n=1}^{\infty} (-1)^n q^n \zeta_3(n).$$