SOME NOTES ON THE ALGEBRA OF AB KMIL SHUJ': A FUSION OF BABYLONIAN AND GREEK ALGEBRA

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A. Theory and practice in the Golden Age of the Arabs.

Abū Kāmil Shuja' (c. 900) was a product of the Golden Age of the Arabs. In this period, the Arabs were more than transmitters of the ancient and Hellenistic knowledge and learning. It is to the credit of the Muslims that they made many solid contributions both in the establishment of new facts and in their utilization. In turn, this higher organization of theoretical investigation and practical learning led eventually to the path of modern scientific methodology.

In chemistry, for example, the Muslims were responsible for the tremendous growth of industrial processes, pharmacy and iatro-chemistry as well as a furtherance of the development of chemical technique and apparatus. Simultaneously, experimental chemistry thrived as it had never done previously. Not only did they maintain their interest in the theoretical aspects of chemical reactions in the laboratory, but the Muslims furthered their practical side. Many Muslim chemical MSS, therefore, often contained labeled drawings of experimental apparatus. Experimental techniques are often described elaborately together with theoretical discussions of the properties of chemical elements and substances and their reactions [3].

It should be noted that the Alexandrian chemists already had contributed much in this direction, no doubt due to the stimulus of the Egyptian and Babylonian practical learning before them. By the time of Zosimos, Greek MSS contain a number of descriptions of operations by diagram of " fusion, calcination, solution, filtration, crystallization, sublimation and especially distillation " [2].

Going back still further, it is of interest that very few written accounts detailing the methods and tools of the Sumerians and Babylonians have been discovered. This is especially true for the more practical sciences of technology such as construction of dwellings, temples or ships, hauling of heavy materials, processing of fibres and weaving of cloth, and so on.

In this climate, the growth of Arabic mathematics paralleled the development of Muslim chemistry. In Abū Kāmil, this fresh approach was made in mathematics. Abū Kāmil utilized the theoretical Greek mathematics without destroying the concrete basis of al-Khwārizmī's algebra [4]. He evolved an algebra born of the practical realities proceeding originally from the Babylonian and then fired in the crucible of Greek theory. The understanding of the necessity of practical principles in Abū Kāmil provides a basis for a sound evolution of algebraic method [5].

Abū Kāmil's text on algebra.

The author has used photocopies of the three MSS extant: Cod. Heb. 225.2, Staatsbibliothek München; Lat. 7377A, Bibliothèque Nat. Paris (cf. L. C. Karpinski, *Bibliotheca Mathematica*, XII, pp. 40-55, 1912); and Cod. Heb. 1029.7, Bibl. Nat. Paris. The latter consists of only the first part of the Algebra. The München MS, the fullest copy of Abū Kāmil's algebra is a Hebrew translation and commentary by Mordecai Finzi, an Italian Jew and member of a noted scholarly family (cf. Carlo

Bernheimer, *La Bibliofilia*, XXVI, pp. 300-25, 1924/25). Through the kindness of Prof. S. Gandz, use has also been made of his autograph copy of a copy made by Dr. Joseph Weinberg who made a German translation, " Die Algebra des Abū Kāmil Šoğa' ben Aṣlam " (München, 1935).

B. The classical equation $x^2 + 21 = 10x$.

1. *Euclid Book II, proposition 5.*

From Euclid, we have the geometric solution of the equation $x^2 + b = ax$. According to the *Commentary of Proclus* (ed. Friedlein, p. 44), this is an ancient proposition and a discovery of the Muse of the Pythagoreans.

> " If a straight lines be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half."
>
> " For let a straight line AB be cut into equal segments at C and into unequal segments at D; I say that the rectangle contained by AD, DB together with the square on CD is equal to the square on CB." [6]

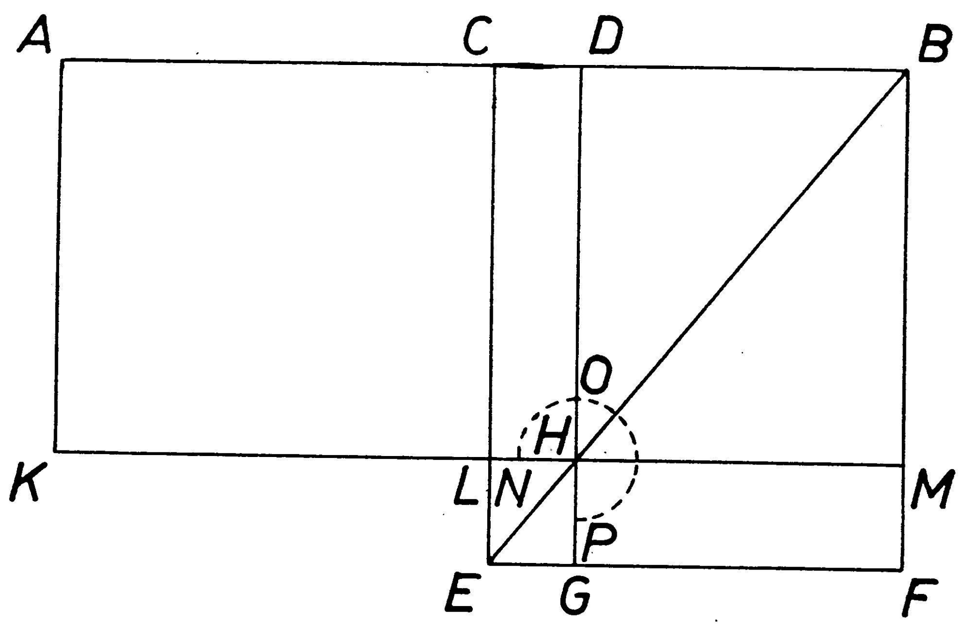

Fig. 1

In the geometric algebra of Euclid, addition and subtraction of simple numbers are, of course, performed by increasing and decreasing the lengths of lines. Multiplication is effected by construction of a rectangle using factors equivalent to the adjacent sides.

2. *Heron's solution.*

Heron proved many of the propositions of Book II by the algebraic method with the use of one line as a figure. The following excerpt is from a later Arabic commentary [1].

> "Then if we wish to demonstrate Heron's proof of this proposition, and the reasoning, we must show that the area outlined by the two parts AD and DB together with the square on line GD is equal to the square on line GB. We take two lines; one of them AD, is divided by point G, and the other line, DB, is not divided. In the proof of proposition 1 of (Book) II, the area that is outlined by the two lines AD and DB is equal to the sum of the two areas, each outlined by line BD with the two divisions AG and GD respectively. Since AG equals GB, then the sum of the two areas, bounded respectively by the two lines GB and BD, and the two lines GD and DB, are equal to the area outlined by the two lines AD and DB. Thus, there remains to us the square on GD. We distribute it as to partners (add it to both sides equally). Then the sum of the two areas bounded by the lines GB and BD, and the lines GD and DB respectively, together with the square on GD is equal to the area outlined by the two lines AD and DB plus the square on GD. But the area that it outlined by the two lines GD and DB plus the square on GD is equal to the area outlined by the two lines BG and GD, from proposition 3 of (Book) II [8]. The sum of the two areas, one outlined by lines BG and GD, and the other by the two lines GB and BD is equal to the area outlined by the two lines AD and DB plus the square on GD. But the demonstration of proposition 2 of (Book) II, the sum of the two areas, outlined respectively by the two lines GB and BD, and the two lines BG and GD, is equal to the square on line GB. The square

on line GB thus is equal to the area that is outlined by the two divisions AD and DB plus the square on GD. This is what we wished to demonstrate."

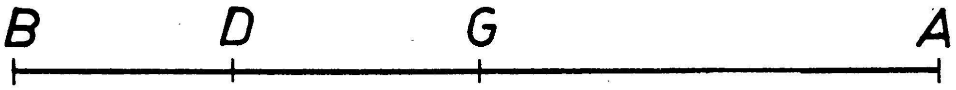

Fig. 1a

In modern symbols, the demonstration of Heron would proceed as follows:

To prove $\overline{AD} \cdot \overline{DB} + \overline{GD}^2 = \overline{GB}^2$.

Given $\overline{AD} = \overline{AG} + \overline{GD}$ and D another point on the line $\overline{AB}$.

By II, 1, $\overline{AD} \cdot \overline{DB} = \overline{BD} \cdot \overline{AG} + \overline{BD} \cdot \overline{GD}$.

But $\overline{AB} = \overline{GB}$ is given.

Then $\overline{GB} \cdot \overline{BD} + \overline{GD} \cdot \overline{DB} = \overline{AD} \cdot \overline{DB}$.

Add GD^2 to both sides of the equation:

$\overline{GB} \cdot \overline{BD} + \overline{GD} \cdot \overline{DB} + \overline{GD}^2 = \overline{AD} \cdot \overline{DB} + \overline{GD}^2$.

But by II, 3, $\overline{GD} \cdot \overline{DB} + \overline{GD}^2 = \overline{BG} \cdot \overline{GD}$.

Hence $\overline{BG} \cdot \overline{GD} + \overline{GD} \cdot \overline{BD} = \overline{AD} \cdot \overline{DB} + \overline{GD}^2$.

But, by II, 2, $\overline{GB} \cdot \overline{BD} + \overline{BG} \cdot \overline{GD} = \overline{GB}^2$.

$\therefore \overline{GB}^2 = \overline{AD} \cdot \overline{DB} + \overline{GD}^2$.

Abū Kāmil does not hesitate to utilize a variation of this procedure at a number of points. For example, he demonstrates its use in his solution of the equations $x + y = 10$, $x/y = 4$ $(x > y)$ [9]; and for $x + y = 10$, $xy = 21$ [10]. In the latter case, Abū Kāmil [11] has the following explanation:

"AG times GB equals twenty one. You divide line AB into two equal parts at point H. Then the product of AG by GB plus HG multiplied by itself equals HB multiplied by itself. The product of BH multiplied by itself is twenty five. AG times GB is twenty one. Then the remainder is HG multiplied by itself, or four; or HG is two. HB is five. Then GB remains as three and AG is seven."

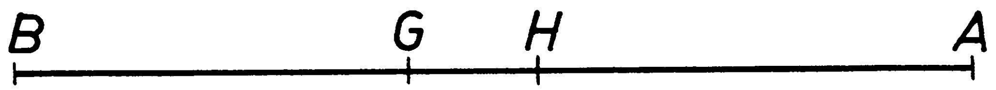

Fig. 2

3. *Al-Khwārizmī's solution.*

Although al-Khwārizmī does not make use of the more abstract one line proof of Heron, nevertheless it is evident that he leans on the concrete concept of root [12] already known in ancient Babylonian times. In his discussion of the equation $x^2 + 21 = 10x$, al-Khwārizmī [13] makes it evident that he is utilizing a concept extremely practical in geometric terms.

> " When a square plus twenty one dirhems are equal to ten roots, we depict the square as a square surface AD of unknown sides. Then we join it to a parallelogram, HB, whose width, HN, is equal to one of the sides of AD. The length of the two surfaces together is equal to the side HC. We know its length to be ten numbers since every square has equal sides and angles; and if one of its sides is multiplied by one, this gives the root of the surface, and if by two, two of its roots. When it is declared that the square plus twenty one equals ten of its roots, we know that the length of the side HC equals ten numbers because the side CD is a root of the square figure. We divide the line CH into two halves on the point G. Then you know that line HG equals line GC, and that line GT equals line CD. Then we extend line GT a distance equal to the difference between line CG and line GT to make the quadrilateral. Then line TK equals line KM, making a quadrilateral MT of equal sides and angles. We know that the line TK and the other sides equals five. Its surface is twenty five obtained by the

multiplication of the half of the roots by itself, or five by five equals twenty five. We know that the surface HB is the twenty one that is added to the square. From the surface HB, we cut off line TK, one of the sides of the surface MT, leaving the surface TA. We take from the line KM line KL which is equal to line GK. We know that line TG equals line ML and that line LK cut from line MK equals line KG. Then the surface MR equals surface TA. We know that surface HT plus surface MR equals surface HB, or twenty one. But surface MT is twenty five. And so, we subtract from surface MT, surface HT and surface MR, both equal to twenty one. We have remaining a small surface, RK, or twenty five less twenty one, or four. Its root, line RG, is equal to line GA, or two. If we subtract it from line CG, which is half of the roots, there remains line AC, or three. This is the root of the first square. If it is added to line GC, which is half of the roots, it comes to seven, or line RC, the root of a larger square. If twenty one is added to it, the result is ten of its roots. This is the figure."

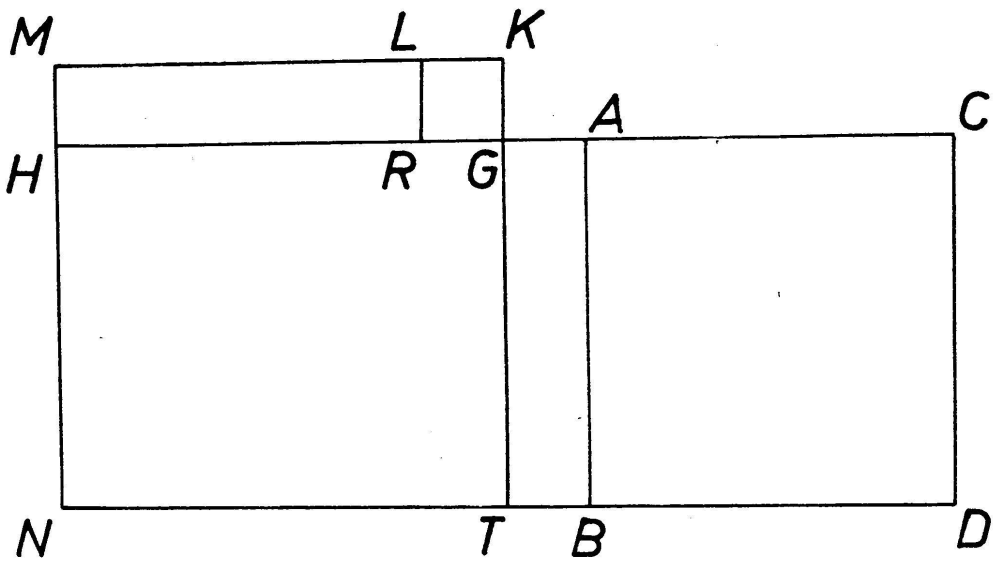

Fig. 3

The algebra of al-Khwārizmī admits the double solution and has a novel manner of utilizing geometry for algebra. Al-Khwārizmī shows the three Arabic types of quadratic equations. In reality, this classification is comparable with the standardized types which had long before been set up by the Babylonians [14]. It is interesting that Al-Khwārizmī shows no evidence of acquaintance with the work of the great Greek algebraist, Diophantus [15].

4. *Abū Kāmil's solution.*

Shūja' [16] also discusses the solution of the question $x^2 + 21 = 10x$, the problem treated by al-Khwārizmī. He solves the equation algebraically in the following steps. Modern symbols have been substituted for the sake of brevity.

$$x = \frac{10}{2} - \sqrt{\left(\frac{10^2}{2}\right) - 21} = 3$$

$$x = \frac{10}{2} + \sqrt{\left(\frac{10}{2}\right)^2 - 21} = 7 .$$

The equation is also solved directly for the two values of x^2:

$$x^2 = \frac{10^2}{2} - 21 - \sqrt{\left(\frac{10^2}{2}\right)^2 - 10^2 . 21} = 9$$

$$x^2 = \frac{10^2}{2} - 21 + \sqrt{\left(\frac{10^2}{2}\right)^2 - 10^2 . 21} = 49 .$$

Then he gives the following demonstration for the equation:

" I shall explain all this. I take the number, twenty one, which is together with the square and larger than the square. I construct the square as a square surface, ABGD, and add to it the twenty one which is the surface ABHL. This surface is greater than surface ABGD. Then, because of this, line BL is greater than line BD. The surface HD equals ten of the roots of ABGD. Then line LD is ten and the surface HB is twenty one, or equal to the product of LB and BD for BD equals BA. Line LD is then divided into two halves by the point X. It had already been divided into two unequal parts by point B. Therefore, the product of LB by BD added to the square on XB is equal to the square on XD. So says Euclid in his book, part two. But the product of line XD by itself is twenty five for its length is five. Line LB times BD is twenty one as has been shown.

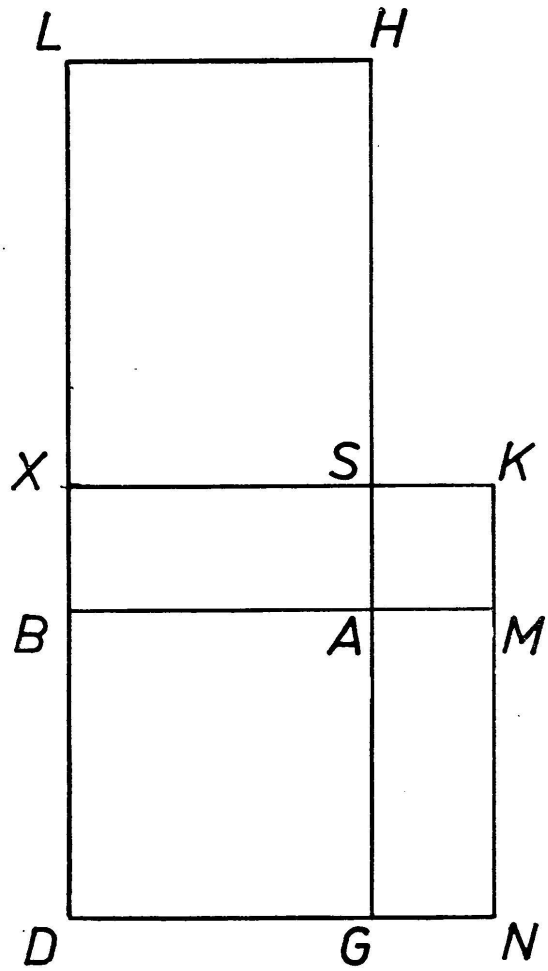

Fig. 4

Then the square on the line XB is four, and its side is two. But line XD is five. Then there remains line BD which equals three. This is the root of the square; the square is nine. If you wish that I prove what I have said, I construct a square on line XD, or surface KD equal to twenty five since line XD is five. Surface XG equals surface XH since line LX equals line XD. Surface AX equals surface AN. Therefore, the three surfaces AX, AD and AN together equal surface HB which is the product of LB by BD, or twenty one. So, surface KA remains equal to four. It is a square since line KN equals line KX, and line XS equals line NM. There remains line KS equal to line KM, equal to two. This is equal to line BX, also equal to two. Then line BD is three. This is the root of the square which is equal to nine.

" I shall explain this question. When you take half of the roots then the result of its multiplication by itself is more than the number, twenty one, that was placed with the square, and less than the square. I set the square as a square surface ABGD and add twenty one, which is the surface ABHW, to it. Surface AD is greater than surface AW as constructed, and line DB is longer than line BW.

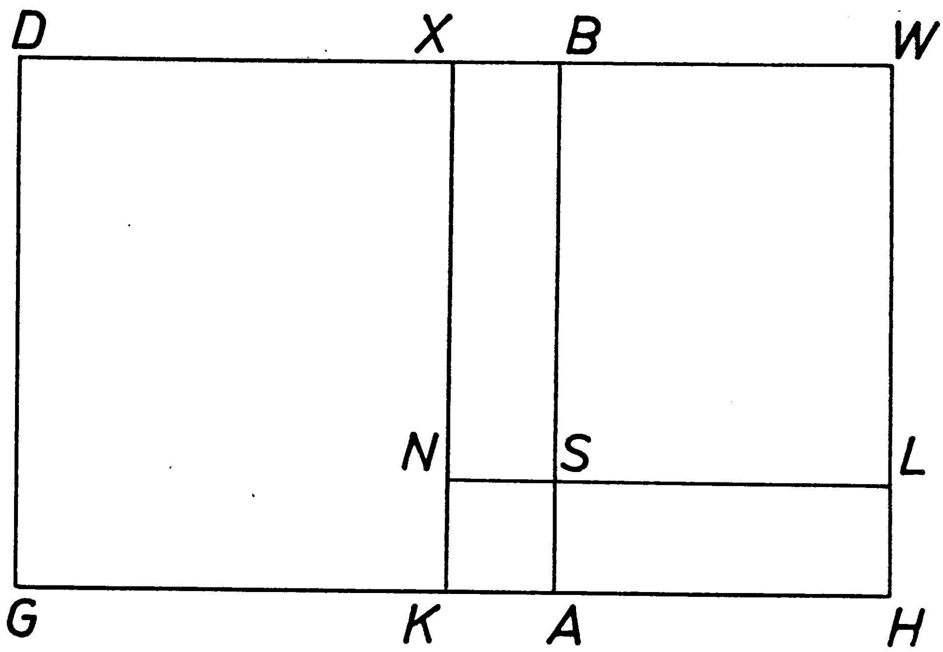

Fig. 5

The surface WG [17] equals ten of the roots of the surface AD. And so line DW is ten. The product of WB [18] by BD is twenty one. You divide line WD into two halves by the point X. Already it is divided into two unequal parts by point B. Thus, the product of WB by BD plus the product of BX by itself equals the product of XD by itself, as in Euclid's book, part two. The product of XD by itself is twenty five. The product of WB by BD is twenty one. There is left the product of line XB by itself, or four. The line XB is the root of four, or two. You add it to line XD which is five. The line BD is then seven, and it is the root of the square; the square is then forty nine. We have explained to you that when we set the square as less than the number, then we get the result by subtraction; when we set it greater than the number, then we get the result by addition.

" If you wish proof of all that we have said, you draw a square surface WN, on the line WX. Extend line XN out to point K. The surface (BN is equal to) surface NH since LN equals line NX, and line KN [19] equals line NS [20] since the surface AN is a square, as we said. Surface WN is twenty five and surface AW is twenty one. Surface AN remains as four and is a square figure since line AB equals line AG, and line KG equals BS since KG equals XD [21] and XD equals XW, and also XW equals XN, and XN equals BS. Therefore BS equals line KG, and line AK remains equal to AS. The surface AN is a square and line SN equals BX, equals two. You add it to line XD which is five to give line BD as seven. This is the root of the square which is forty nine."

Abū Kāmil then goes on with a geometric discussion of the equation $x^2 + 25 = 10x$, a special case where the square equals the number and the root of the square is equal to half of the root on the right side of the equation.

In Book II, Euclid has geometric demonstrations of algebraic formulas while, on the other hand, the works of the above mentioned Muslims are primarily algebraic with geometric explanations. It has been shown already that Greek geometry and algebra had no direct influence upon al-Khwārizmī [22]. The fact that geometric algebra is found in Euclid in such seemingly different form would tend to strengthen this idea. Moreover, on closer examination of Euclid, we find that " . . . the proofs of all the first ten propositions of Book II are practically independent of each other . . ." Heath then asks and answers the question, " what then was Euclid's intention, first in inserting some propositions not immediately required, and secondly in making the proofs of the first ten independent of each other ?

surely the object was to show the power of the method of geometrical algebra as much as to arrive at results" [23].

With the Babylonian accent on the algebraic form of geometry and the ensuing dependence of al-Khwārizmī upon this source, the latter's form of geometric algebra is fully expected. Thus, from the works of al-Khwārizmī and both Heron and Euclid, respectively representing the Babylonian and Greek forms of algebra, Abū Kāmil presented algebra on a unique level. This admits of theoretical explanation and demonstration, and provides the means of integrating Babylonian practice with Greek theory into a more virile approach [24].

C. Other examples of Abū Kāmil's methodology.

Abū Kāmil was the earliest algebraist to work out the solutions directly for the square of the unknown. In the problem quoted above he makes use of the following solutions:

$$x^2 = \frac{b^2}{2} - c - \sqrt{\left(\frac{c^2}{2}\right)^2 - b^2 c}$$

and for the second value

$$x^2 = \frac{b^2}{2} - c + \sqrt{\left(\frac{c^2}{2}\right)^2 - b^2 c}$$

The addition and substraction of [25] radicals was effected rhetorically by means of the relation now known as

$$\sqrt{a} \pm \sqrt{b} = \sqrt{a + b \pm 2\sqrt{a\,b}}$$

An example of this is given in $\sqrt{9} - \sqrt{4}$, whose solution is determined to be $\sqrt{9 + 4 - 2\sqrt{36}} = 1$ in the following [26]:

" On subtraction of roots from each other.

" When you wish to subtract (the root of) four from the root of nine so that the difference of the roots be another number, you add nine to four to give thirteen [27]. Then multiply nine by four to give thirty six [28]. Take two roots of it to give twelve. You subtract it from thirteen to get one. The root of one is the difference between the root of nine and the root of four. It is one. I shall explain it to you by this figure:

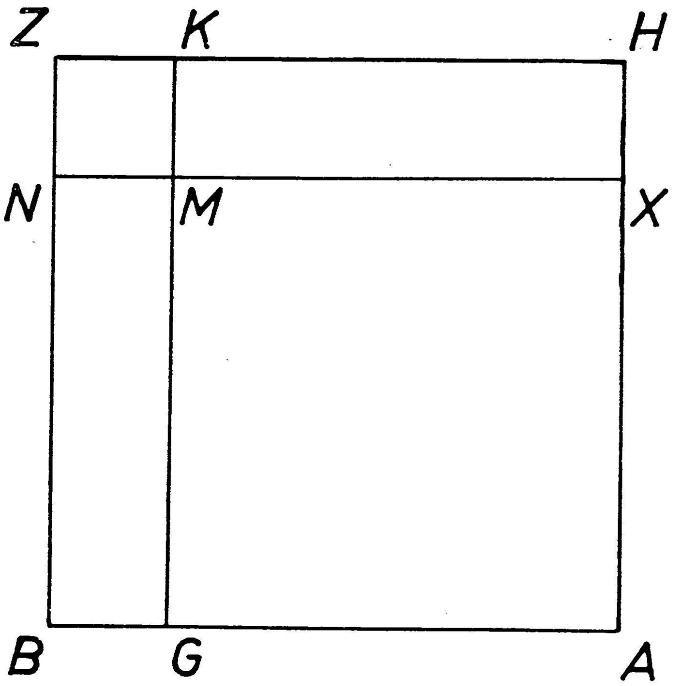

Fig.6

"We construct the line AB as the root of nine and the line AG as the root of four. When we subtract line AG from line AB, line GB remains. When we wish to know the value of line GB as a root, we construct on line AG [29], the square surface AM, which is four. You extend line XM to N and line GM to K. One knows that the square MZ is the product of GB by itself. Surface XK is two since the entire surface AK is six, or the product of line AG, which is the root of four, by AH which is the root of nine. Surface AM [30] is four and so surface XK remains as two. Also surface MB is two. Square MZ remains then as one and line MN is its root, or one. Line MN is equal to line GB and so it is demonstrated."

The use of this formula for the addition and subtraction of radicals is found in the later works of al-Karkhī [31] and Leonardo Fibonacci [32]. In Abū Kāmil, there can be no doubt that Book X of Euclid influenced him to introduce the irrational as a solution for some of his quadratic equations.

Abū Kāmil was the first Muslim algebraist to work with powers of the unknown higher than the square. In his algebra he uses the second, third, fourth, fifth, sixth and eighth powers of x. The names of these higher powers are based on the addition of exponents as we know them today. The development of this method of reckoning with exponents did not progress in a straight line. Hundred of years later symbols were still being used which were based on the systems of exponent multiplication [33].

D. Fusion of Babylonian and Greek Algebra.

From the passages of Abū Kāmil quoted above, it is evident that he was influenced by traditions which ultimately may be traced back to Babylonian and Greek Sources. On the one hand it is a further development of al-Khwārizmī's method, originally Babylonian, and on the other a utilization of the best algebraic innovations of the Greeks. It is possible that the latter were known, in part, to Abū Kāmil, through the works of Heron. The influence of Heron has already been established in the case of the great Hebrew geometer, Abraham Savasorda (12th century), as seen in his *Encyclopedia* [34]. It is interesting that Savasorda [35] who pioneered a new approach to geometry and Abū Kāmil who did the same for algebra should both have been influenced, directly or indirectly, by the great Alexandrian, Heron.

In turn, Abū Kāmil, as did Savasorda, exerted great influence upon al-Karkhī [36] and Leonardo Fibonacci, both of whom made use of many problems found in Abū Kāmil's algebra. In spite of the fact that Leonardo had squeezed Abū Kāmil's algebra dry of almost all his examples, nevertheless, enough material remained in the text so that Mordecai Finzi of the fifteenth century deemed it worthwhile to translate it into Hebrew and to insert further comments of his own.

In the painful growth of the integration of mathematical abstraction with its counterpart, the schematization and understanding of the practical basis, we have the seed of the forward development of mathematical science. With Abū Kāmil, mathematical abstraction attained recognition, not for its own sake, but because of its value when properly integrated with a more practical mathematical methodology.

NOTES AND REFERENCES

[1] The author is indebted to Temple University for a research grant which aided in the preparation of this paper.

[2] Cf. M. LEVEY, "Beginnings of Early Chemical Equipment: Some Apparatus of Ancient Mesopotamia", *J. Chem. Educ.*, 32, 180-184 (1955); "Evidences of Ancient Distillation, Sublimation and Extraction in Mesopotamia", *Centaurus*, IV, 23-33 (1955).

[3] An excellent example of this may be found in Maqbūl Aḥmad, "A Persian Translation of the Eleventh Century Arabic Alchemical Treatise 'Ain Aṣ-ṣan'ah Wa 'Aun Aṣ-ṣana'ah." (*Mem. As. Soc. Bengal,* VIII, No. 7, pp. 419-460 (1929); STAPLETON, Azo and ḤUSAIN, "Chemistry in 'Iraq and Persia in the Tenth Century A. D.", *idem*, VIII, No. 6, pp. 317-418 (1927); also STAPLETON, "Alchemical Equipment in the Eleventh Century, A.D.", *idem*, Vol. I, No. 4, pp. 47-70 (1905) for apparatus used and its implication for the importance of experimental work in chemistry (p. 65 bot.).

[4] Almost half of al-Khwārizmī's *Algebra* is devoted to problems arising directly from the inheritance laws of the Muslims.

[5] Alfred North WHITEHEAD, "Science and the Modern World", p. 48, gives another viewpoint, "... the utmost abstractions are the true weapons with which to control our thought of concrete fact".

[6] Thomas L. HEATH, "The Thirteen Books of Euclid's Elements", I, p. 382 ff. (1908).

[7] BESTHORN and HEIBERG, "Codex Leidensis 399, 1. Euclidis Elementa Ex Interpretatione Al-Hadschdschadschii Cum Commentariis Al-Narizii", Partis II, Fasc. I (Hauniae: 1900), pp. 27, 29. Translated by the present author from the Arabic with errors corrected. Corrected figure is given.

[8] BESTHORN and HEIBERG, p. 29, incorrectly have, "Sed ex II, 2 summa duorum spatiorum ..." The figure for this demonstration is incorrectly given. G should be placed in the center of Line AB as a step of the proof would require.

[9] Cod. Heb. 225-2, Staatsbibliothek München, fol. 112*a*.

[10] München Cod. Heb. 225.2, fol. 113*a*.

[11] *Ibid.*, fols. 113*a*, 113*b*.

[12] For a full discussion of root and its practical significance, cf. S. GANDZ, "The Origin of the Term 'Root'", *Am. Math. Monthly, 33*, pp. 261-5 (1926); *35*, pp. 67-75 (1928). Cf. also Solŏmon GANDZ, "The Mishnat Ha Middot, The First Hebrew Geometry of About 150 C.E.", in Quellen und Studien zur Geschichte der Math., A. Quellen, II (Berlin, 1932), *passim*. The term "root" in most cases meant "square basis", that by whose multiplication we get the square area.

[13] More literal translation by the author from the Arabic edited by F. ROSEN, "The Algebra of Mohammed Ben Musa" (London,

1831), pp. 11, 12, 13 Arabic, checked with the MS photostat. Cf. also Rosen's translation, pp. 16, 17, 18 Eng.

[14] S. GANDZ, The Origin and Development of Quadratic Equations in Babylonian, Greek and Early Arabic Algebra, *Osiris III*, p. 542 (1937).

[15] S. GANDZ, The Sources of Al-Khwārizmı's Algebra, *Osiris*, I, pp. 268-9 (1936).

[16] München Cod. Heb. 225.2, fols. 98*b*, 99*a*, 99*b*. The Hebrew has been translated anew because of difficulties found in the German translation of Weinberg. Further, the translation has been redone to recapture the original flavor and approach of the Hebrew version. Enough of this passage has been given so that it may be compared with the corresponding section in al-Khwārizmī, and those of Heron and Euclid.

[17] Text incorrectly reads XG.

[18] Text incorrectly reads BB.

[19] Corrected from KX in Josef WEINBERG, *op. cit.*, p. 27.

[20] Corrected from NB.

[21] Weinberg here omitted a line in his translation, *op. cit.*, p. 28.

[22] S. GANDZ, *Osiris*, I, p. 263 (1936).

[23] HEATH, *op. cit.*, I, p. 377.

[24] Cf. also Abū Kāmil's treatise on the pentagon and decagon for their algebraic treatment. Gustavo SACERDOTE, Il trattato del pentagono e del decagono, " Festschrift... Moritz Steinschneider's " (Leipzig: 1897), pp. 169-194.

[25] Abū Kāmil's algebra, as compared with al-Khwārizmī's presentation, has a very full description of fundamental algebraic operations. Cf. the elaborate and carefully ordered series of problems dealing with irrationals in Simon Motot (G. SACERDOTE, transl., *Revue Etudes juives*, 1893/4). Also similar passages to that in Abū Kāmil found in al-Karkhī (F. WOEPCKE, " Extrait du Fakhrî " (Paris, 1853), pp. 57-9) and in Leonardo FIBONACCI (Boncompagni, ed., " Scritti di Leonardo Pisano ", I, pp. 363-5.

[26] München Cod. Heb. 225.2, fol. 110*a*.

[27] Text has sixteen.

[28] Text has thirty two.

[29] Text has AB.

[30] Text has AX.

[31] WOEPCKE, *op. cit.*

[32] LEONARDO, *op. cit.*

[33] Cf. the sixteenth century algebra edited by M. CURTZE, " Die Algebra des Initius Algebras as Ylem geometram magistrum suum ", *Urkunden z. Gesch. d. Math. (Abhandl. z. Gesch. d. Math. Wiss.)*, XIII (1902).

[34] *Vide* by the present author, " Abraham Savasorda and his Algorism: A Study in Early European Logistic ", *Osiris*, XI (1954).

[35] Cf. Martin LEVEY, The Encyclopedia of Abraham Savasorda: " A Departure in Mathematical Methodology ", *Isis*, 43, pp. 257-64 (1952).

[36] Johannes TROPFKE, " Gesch. d. Elementar-Mathematik " (Berlin: 1937), Vol. I, p. 108.