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Bernheimer, *La Bibliofilia*, XXVI, pp. 300-25, 1924/25). Through the kindness of Prof. S. Gandz, use has also been made of his autograph copy of a copy made by Dr. Joseph Weinberg who made a German translation, " Die Algebra des Abū Kāmil Šoğa' ben Aṣlam " (München, 1935).

B. The classical equation $x^2 + 21 = 10x$.

1. *Euclid Book II, proposition 5.*

From Euclid, we have the geometric solution of the equation $x^2 + b = ax$. According to the *Commentary of Proclus* (ed. Friedlein, p. 44), this is an ancient proposition and a discovery of the Muse of the Pythagoreans.

" If a straight lines be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half."

" For let a straight line AB be cut into equal segments at C and into unequal segments at D; I say that the rectangle contained by AD, DB together with the square on CD is equal to the square on CB." [6]

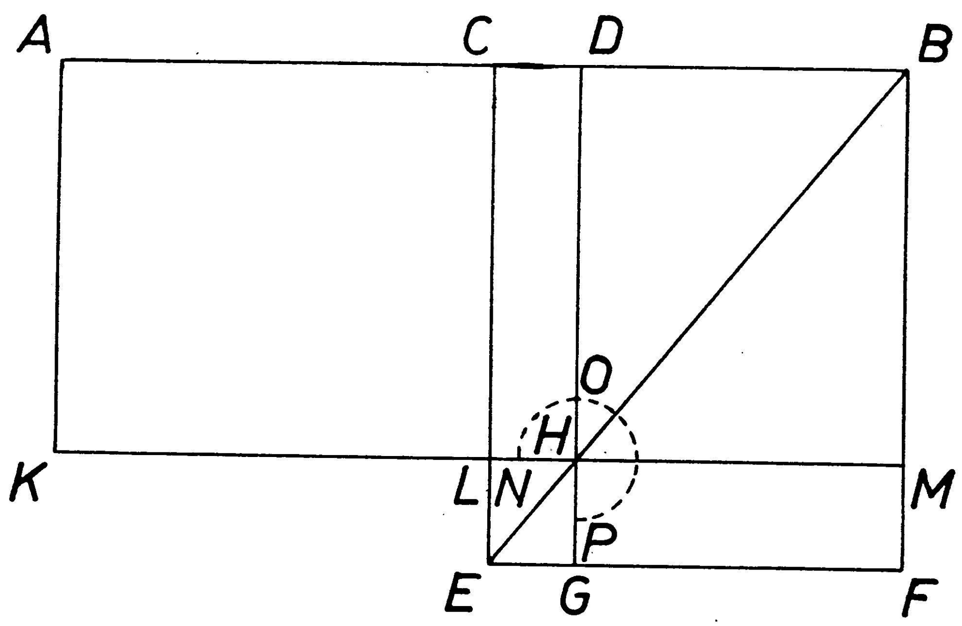

Fig. 1

In the geometric algebra of Euclid, addition and subtraction of simple numbers are, of course, performed by increasing and decreasing the lengths of lines. Multiplication is effected by construction of a rectangle using factors equivalent to the adjacent sides.

2. *Heron's solution.*

Heron proved many of the propositions of Book II by the algebraic method with the use of one line as a figure. The following excerpt is from a later Arabic commentary [1].

> "Then if we wish to demonstrate Heron's proof of this proposition, and the reasoning, we must show that the area outlined by the two parts AD and DB together with the square on line GD is equal to the square on line GB. We take two lines; one of them AD, is divided by point G, and the other line, DB, is not divided. In the proof of proposition 1 of (Book) II, the area that is outlined by the two lines AD and DB is equal to the sum of the two areas, each outlined by line BD with the two divisions AG and GD respectively. Since AG equals GB, then the sum of the two areas, bounded respectively by the two lines GB and BD, and the two lines GD and DB, are equal to the area outlined by the two lines AD and DB. Thus, there remains to us the square on GD. We distribute it as to partners (add it to both sides equally). Then the sum of the two areas bounded by the lines GB and BD, and the lines GD and DB respectively, together with the square on GD is equal to the area outlined by the two lines AD and DB plus the square on GD. But the area that it outlined by the two lines GD and DB plus the square on GD is equal to the area outlined by the two lines BG and GD, from proposition 3 of (Book) II [8]. The sum of the two areas, one outlined by lines BG and GD, and the other by the two lines GB and BD is equal to the area outlined by the two lines AD and DB plus the square on GD. But the demonstration of proposition 2 of (Book) II, the sum of the two areas, outlined respectively by the two lines GB and BD, and the two lines BG and GD, is equal to the square on line GB. The square

on line GB thus is equal to the area that is outlined by the two divisions AD and DB plus the square on GD. This is what we wished to demonstrate."

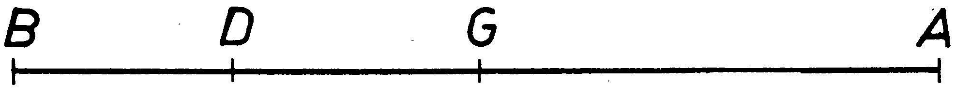

Fig. 1a

In modern symbols, the demonstration of Heron would proceed as follows:

To prove $\overline{AD} \cdot \overline{DB} + \overline{GD}^2 = \overline{GB}^2$.

Given $\overline{AD} = \overline{AG} + \overline{GD}$ and D another point on the line $\overline{AB}$.

By II, 1, $\overline{AD} \cdot \overline{DB} = \overline{BD} \cdot \overline{AG} + \overline{BD} \cdot \overline{GD}$.

But $\overline{AB} = \overline{GB}$ is given.

Then $\overline{GB} \cdot \overline{BD} + \overline{GD} \cdot \overline{DB} = \overline{AD} \cdot \overline{DB}$.

Add GD^2 to both sides of the equation:

$\overline{GB} \cdot \overline{BD} + \overline{GD} \cdot \overline{DB} + \overline{GD}^2 = \overline{AD} \cdot \overline{DB} + \overline{GD}^2$.

But by II, 3, $\overline{GD} \cdot \overline{DB} + \overline{GD}^2 = \overline{BG} \cdot \overline{GD}$.

Hence $\overline{BG} \cdot \overline{GD} + \overline{GD} \cdot \overline{BD} = \overline{AD} \cdot \overline{DB} + \overline{GD}^2$.

But, by II, 2, $\overline{GB} \cdot \overline{BD} + \overline{BG} \cdot \overline{GD} = \overline{GB}^2$.

$\therefore \overline{GB}^2 = \overline{AD} \cdot \overline{DB} + \overline{GD}^2$.

Abū Kāmil does not hesitate to utilize a variation of this procedure at a number of points. For example, he demonstrates its use in his solution of the equations $x + y = 10$, $x/y = 4$ $(x > y)$ [9]; and for $x + y = 10$, $xy = 21$ [10]. In the latter case, Abū Kāmil [11] has the following explanation:

" AG times GB equals twenty one. You divide line AB into two equal parts at point H. Then the product of AG by GB plus HG multiplied by itself equals HB multiplied by itself. The product of BH multiplied by itself is twenty five. AG times GB is twenty one. Then the remainder is HG multiplied by itself, or four; or HG is two. HB is five. Then GB remains as three and AG is seven."

Fig. 2

3. *Al-Khwārizmī's solution.*

Although al-Khwārizmī does not make use of the more abstract one line proof of Heron, nevertheless it is evident that he leans on the concrete concept of root [12] already known in ancient Babylonian times. In his discussion of the equation $x^2 + 21 = 10x$, al-Khwārizmī [13] makes it evident that he is utilizing a concept extremely practical in geometric terms.

> " When a square plus twenty one dirhems are equal to ten roots, we depict the square as a square surface AD of unknown sides. Then we join it to a parallelogram, HB, whose width, HN, is equal to one of the sides of AD. The length of the two surfaces together is equal to the side HC. We know its length to be ten numbers since every square has equal sides and angles; and if one of its sides is multiplied by one, this gives the root of the surface, and if by two, two of its roots. When it is declared that the square plus twenty one equals ten of its roots, we know that the length of the side HC equals ten numbers because the side CD is a root of the square figure. We divide the line CH into two halves on the point G. Then you know that line HG equals line GC, and that line GT equals line CD. Then we extend line GT a distance equal to the difference between line CG and line GT to make the quadrilateral. Then line TK equals line KM, making a quadrilateral MT of equal sides and angles. We know that the line TK and the other sides equals five. Its surface is twenty five obtained by the

multiplication of the half of the roots by itself, or five by five equals twenty five. We know that the surface HB is the twenty one that is added to the square. From the surface HB, we cut off line TK, one of the sides of the surface MT, leaving the surface TA. We take from the line KM line KL which is equal to line GK. We know that line TG equals line ML and that line LK cut from line MK equals line KG. Then the surface MR equals surface TA. We know that surface HT plus surface MR equals surface HB, or twenty one. But surface MT is twenty five. And so, we subtract from surface MT, surface HT and surface MR, both equal to twenty one. We have remaining a small surface, RK, or twenty five less twenty one, or four. Its root, line RG, is equal to line GA, or two. If we subtract it from line CG, which is half of the roots, there remains line AC, or three. This is the root of the first square. If it is added to line GC, which is half of the roots, it comes to seven, or line RC, the root of a larger square. If twenty one is added to it, the result is ten of its roots. This is the figure."

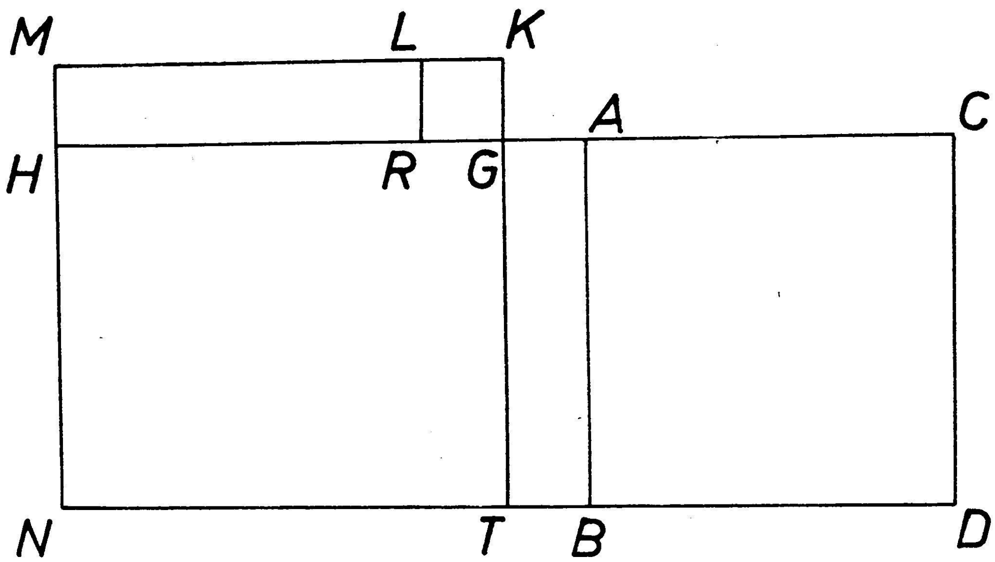

Fig. 3

The algebra of al-Khwārizmī admits the double solution and has a novel manner of utilizing geometry for algebra. Al-Khwārizmī shows the three Arabic types of quadratic equations. In reality, this classification is comparable with the standardized types which had long before been set up by the Babylonians [14]. It is interesting that Al-Khwārizmī shows no evidence of acquaintance with the work of the great Greek algebraist, Diophantus [15].

4. *Abū Kāmil's solution.*

Shūja' [16] also discusses the solution of the question $x^2 + 21 = 10x$, the problem treated by al-Khwārizmī. He solves the equation algebraically in the following steps. Modern symbols have been substituted for the sake of brevity.

$$x = \frac{10}{2} - \sqrt{\left(\frac{10^2}{2}\right) - 21} = 3$$

$$x = \frac{10}{2} + \sqrt{\left(\frac{10}{2}\right)^2 - 21} = 7 .$$

The equation is also solved directly for the two values of x^2:

$$x^2 = \frac{10^2}{2} - 21 - \sqrt{\left(\frac{10^2}{2}\right)^2 - 10^2 . 21} = 9$$

$$x^2 = \frac{10^2}{2} - 21 + \sqrt{\left(\frac{10^2}{2}\right)^2 - 10^2 . 21} = 49 .$$

Then he gives the following demonstration for the equation:

" I shall explain all this. I take the number, twenty one, which is together with the square and larger than the square. I construct the square as a square surface, ABGD, and add to it the twenty one which is the surface ABHL. This surface is greater than surface ABGD. Then, because of this, line BL is greater than line BD. The surface HD equals ten of the roots of ABGD. Then line LD is ten and the surface HB is twenty one, or equal to the product of LB and BD for BD equals BA. Line LD is then divided into two halves by the point X. It had already been divided into two unequal parts by point B. Therefore, the product of LB by BD added to the square on XB is equal to the square on XD. So says Euclid in his book, part two. But the product of line XD by itself is twenty five for its length is five. Line LB times BD is twenty one as has been shown.

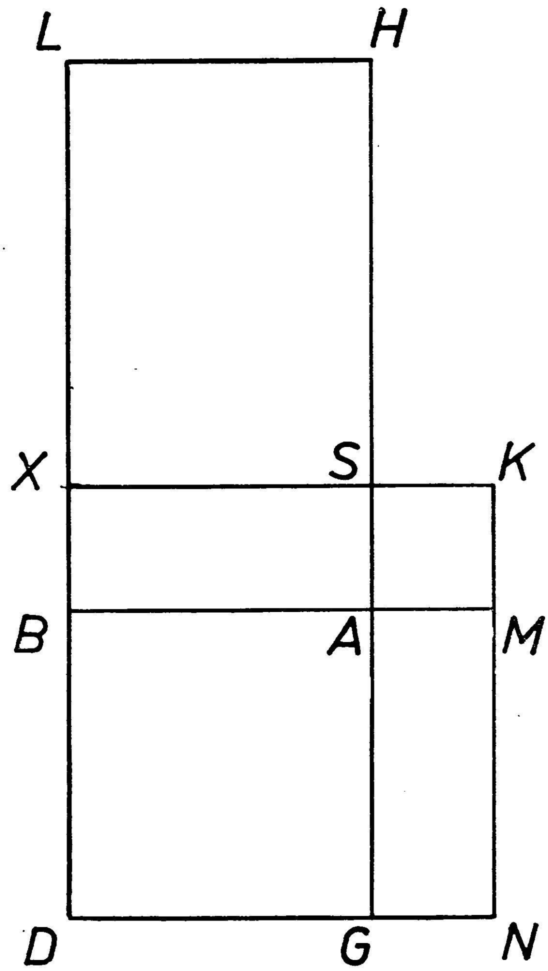

Fig. 4

Then the square on the line XB is four, and its side is two. But line XD is five. Then there remains line BD which equals three. This is the root of the square; the square is nine. If you wish that I prove what I have said, I construct a square on line XD, or surface KD equal to twenty five since line XD is five. Surface XG equals surface XH since line LX equals line XD. Surface AX equals surface AN. Therefore, the three surfaces AX, AD and AN together equal surface HB which is the product of LB by BD, or twenty one. So, surface KA remains equal to four. It is a square since line KN equals line KX, and line XS equals line NM. There remains line KS equal to line KM, equal to two. This is equal to line BX, also equal to two. Then line BD is three. This is the root of the square which is equal to nine.

" I shall explain this question. When you take half of the roots then the result of its multiplication by itself is more than the number, twenty one, that was placed with the square, and less than the square. I set the square as a square surface ABGD and add twenty one, which is the surface ABHW, to it. Surface AD is greater than surface AW as constructed, and line DB is longer than line BW.

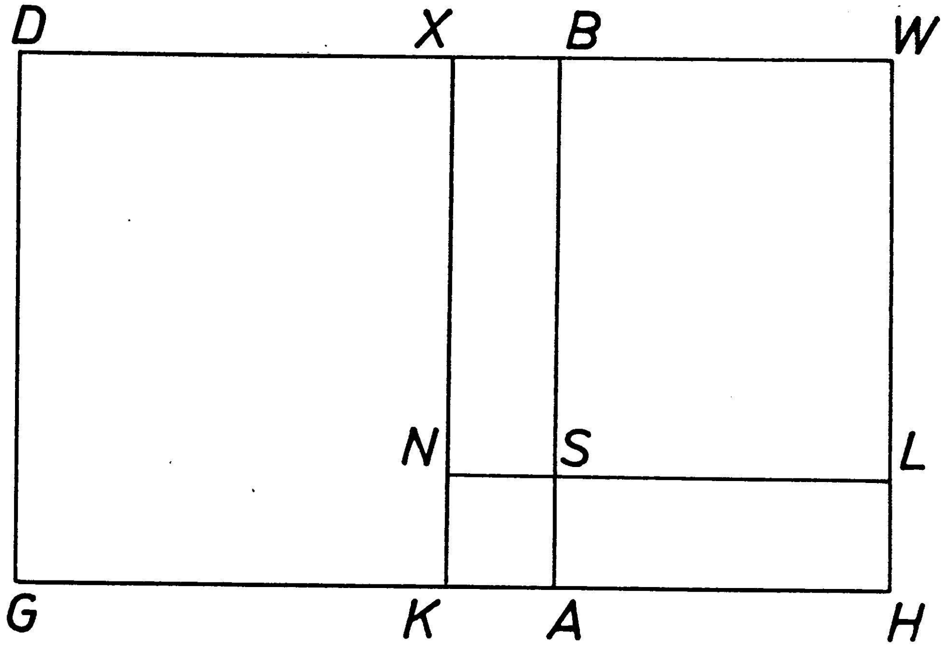

Fig. 5

The surface WG [17] equals ten of the roots of the surface AD. And so line DW is ten. The product of WB [18] by BD is twenty one. You divide line WD into two halves by the point X. Already it is divided into two unequal parts by point B. Thus, the product of WB by BD plus the product of BX by itself equals the product of XD by itself, as in Euclid's book, part two. The product of XD by itself is twenty five. The product of WB by BD is twenty one. There is left the product of line XB by itself, or four. The line XB is the root of four, or two. You add it to line XD which is five. The line BD is then seven, and it is the root of the square; the square is then forty nine. We have explained to you that when we set the square as less than the number, then we get the result by subtraction; when we set it greater than the number, then we get the result by addition.

" If you wish proof of all that we have said, you draw a square surface WN, on the line WX. Extend line XN out to point K. The surface (BN is equal to) surface NH since LN equals line NX, and line KN [19] equals line NS [20] since the surface AN is a square, as we said. Surface WN is twenty five and surface AW is twenty one. Surface AN remains as four and is a square figure since line AB equals line AG, and line KG equals BS since KG equals XD [21] and XD equals XW, and also XW equals XN, and XN equals BS. Therefore BS equals line KG, and line AK remains equal to AS. The surface AN is a square and line SN equals BX, equals two. You add it to line XD which is five to give line BD as seven. This is the root of the square which is forty nine."

Abū Kāmil then goes on with a geometric discussion of the equation $x^2 + 25 = 10x$, a special case where the square equals the number and the root of the square is equal to half of the root on the right side of the equation.

In Book II, Euclid has geometric demonstrations of algebraic formulas while, on the other hand, the works of the above mentioned Muslims are primarily algebraic with geometric explanations. It has been shown already that Greek geometry and algebra had no direct influence upon al-Khwārizmī [22]. The fact that geometric algebra is found in Euclid in such seemingly different form would tend to strengthen this idea. Moreover, on closer examination of Euclid, we find that " . . . the proofs of all the first ten propositions of Book II are practically independent of each other . . ." Heath then asks and answers the question, " what then was Euclid's intention, first in inserting some propositions not immediately required, and secondly in making the proofs of the first ten independent of each other ?

surely the object was to show the power of the method of geometrical algebra as much as to arrive at results" [23].

With the Babylonian accent on the algebraic form of geometry and the ensuing dependence of al-Khwārizmī upon this source, the latter's form of geometric algebra is fully expected. Thus, from the works of al-Khwārizmī and both Heron and Euclid, respectively representing the Babylonian and Greek forms of algebra, Abū Kāmil presented algebra on a unique level. This admits of theoretical explanation and demonstration, and provides the means of integrating Babylonian practice with Greek theory into a more virile approach [24].

C. Other examples of Abū Kāmil's methodology.

Abū Kāmil was the earliest algebraist to work out the solutions directly for the square of the unknown. In the problem quoted above he makes use of the following solutions:

$$x^2 = \frac{b^2}{2} - c - \sqrt{\left(\frac{c^2}{2}\right)^2 - b^2 c}$$

and for the second value

$$x^2 = \frac{b^2}{2} - c + \sqrt{\left(\frac{c^2}{2}\right)^2 - b^2 c}$$

The addition and substraction of [25] radicals was effected rhetorically by means of the relation now known as

$$\sqrt{a} \pm \sqrt{b} = \sqrt{a + b \pm 2\sqrt{a\,b}}$$

An example of this is given in $\sqrt{9} - \sqrt{4}$, whose solution is determined to be $\sqrt{9 + 4 - 2\sqrt{36}} = 1$ in the following [26]:

> " On subtraction of roots from each other.
>
> " When you wish to subtract (the root of) four from the root of nine so that the difference of the roots be another number, you add nine to four to give thirteen [27]. Then multiply nine by four to give thirty six [28]. Take two roots of it to give twelve. You subtract it from thirteen to get one. The root of one is the difference between the root of nine and