1. THE DEFINITION OF NORM

THE NORM OF A REAL LINEAR TRANSFORMATION IN MINKOWSKI SPACE

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(Reçu le 27 février 1958)

1. The definition of norm

By the Minkowski space $l^p(n)$ we mean the space of vectors $x = (\xi_1, \ldots, \xi_n)$ with the norm of x defined by

$$\| x \|_p = \Big(\sum_i | \xi_i |^p \Big)^{1/p} .$$

Here it is supposed that $p \geqq 1$, so that $\| x \|_p$ is a norm on $l^p(n)$.

If $l^p(n)$ and $l^q(m)$ are Minkowski spaces of dimensions n and m, respectively, a linear transformation A of $l^p(n)$ into $l^q(m)$ is determined by a matrix (a_{jk}) of constants ($j = 1, \ldots, m$, $k = 1, \ldots, n$); if A transforms x into $y = (\eta_1, \ldots, \eta_m)$, the η's are given in terms of the ξ's by the equations

$$\eta_j = \sum_{k=1}^{n} a_{jk} \xi_k \qquad j = 1, \ldots, m .$$

If we write $y = \mathrm{A}x$, the norm of A is defined as

$$\| \mathrm{A} \| = \max_{x \neq 0} \frac{\| \mathrm{A}x \|_q}{\| x \|_p} = \max_{x \neq 0} \frac{\Big(\sum_j \Big| \sum_k a_{jk} \xi_k \Big|^q \Big)^{1/q}}{\Big(\sum_k | \xi_k |^p \Big)^{1/p}} .$$

We may consider all of these things with respect to the complex field, letting the vector components $\xi_1, \ldots, \xi_n$ and the matrix elements a_{jk} be complex numbers. In this case we call $l^p(n)$ a *complex* Minkowski space and A a *complex* linear transformation. But we may equally well confine our attention to real scalars, in which case the space and the transformation are called *real*.

Now, if A is a transformation determined by a matrix of real elements a_{jk}, the transformation can be considered either as a real transformation or as a complex transformation, and accordingly there are two possible definitions of its norm. If in (1) we allow x to vary over all nonzero elements of the complex space $l^p(n)$, we get the norm of A as a complex transformation, whereas if we restrict the vector x to have real components, we get the norm of A as a real transformation. Let us denote these two norms by

$$\| A \|_c \quad \text{and} \quad \| A \|_r .$$

2. The theorem

We shall prove the following result:

Theorem *Let* A *be a transformation of* $l^p(n)$ *into* $l^q(m)$ *determined by a matrix* (a_{jk}) *of real constants. Suppose* $q \geqq p \geqq 1$. *Then*

$$\| A \|_c = \| A \|_r . \tag{2}$$

Proof. We first observe that when p is fixed and q varies subject to $q \geqq p$, $\| A \|$ is a continuous function of q at $q = p$, regardless of whether we have a real or a complex transformation. For, let the dependence on q be exhibited by writing $\|A\| = M(q)$. It is well known that $\| y \|_q$ does not decrease as q decreases. Hence $p < q$ implies

$$\frac{\| Ax \|_q}{\| x \|_p} \leqq \frac{\| Ax \|_p}{\| x \|_p},$$

whence also $M(q) \leqq M(p)$. Now suppose that x is chosen $(x \neq 0)$ so that

$$\frac{\| Ax \|_p}{\| x \|_p} = M(p) .$$