VECTOR FIELDS ON SPHERES AND ALLIED PROBLEMS

VECTOR FIELDS ON SPHERES AND ALLIED PROBLEMS [1])

by Raoul BOTT [2])

The problem on whose development I would like to report is very easily stated.

A set of k maps $f_i: E_n \to E_n$, $i = 1, \ldots, k$; of Euclidean n-space into itself will be called an orthogonal k-system if:

$$f_1(x) = x \quad \textit{for all} \quad x \in E_n\,, \quad \text{and}$$

$f_1(x), \ldots, f_k(x)$ *form an orthonormal system whenever* x *is a unit vector.*

With this terminology our question is the following one:

Find the greatest integer k, *so that* E_n *admits an orthogonal* k-*system.*

Geometrically an orthogonal k-system on E_n is precisely a continuous $(k - 1)$ frame on the unit sphere $S_{n-1} \subset E_n$, as is seen immediately once the tangent space of S_{n-1} at $x \in S_{n-1}$ is identified with the orthogonal complement to the subspace generated by x. We are therefore dealing with a very special case of the general question of how many independent vector fields exist on a manifold, and this central position of our question has made it the favorite testing ground of progress in Algebraic Topology. It is not in the spirit of this talk to recount the precise evolution of the problem, or pay tribute to the many people who have contributed to it, be it through the general theory of vector fields, or through a specific attack—Poincaré, Hopf, Stiefel, Whitney, Steenrod, Whitehead, Wu, Addem, are just a few names which come to mind—rather, I would first like to recall an old algebraic result in this direction and then go on to some of the most recent work which topologically confirms the algebraic findings.

---

1) Talk delivered at the Zurich Colloquium on Differential Geometry and Topology, June 1960.

2) The Author holds an A. P. Sloan Fellowship.

First of all it is convenient to recast the question in this manner:

PROBLEM I. *Given* k, *for what* n *does* E_n *admit an orthogonal* k-*system ?*

Before discussing this question, let us formulate its linear version. If we call an orthogonal k-system linear whenever each of the functions f_i, $i = 1, \ldots, k$; comprising it is linear, then this querry is:

For which n, *does* E_n *admit a linear orthogonal* k-*system ?*

This purely algebraic question can also be expressed as follows. A linear map

$$\mu : E_k \otimes E_n \to E_n \qquad k \leqq n$$

will be said to define E_n as an E_k-module if for all x in E_k and y in E_n

$$|\mu(x \otimes y)| = |x| \cdot |y| ,$$

The vertical bars denoting the Euclidean norm. It is then quite easy to check that E_n admits the structure of an E_k-module if and only if E_n admits a linear orthogonal k-system. In this guise then, the associated algebraic problem is stated as follows:

PROBLEM I^L. *What are the dimensions of the possible* E_k-*modules.*

The complete solution of the linear problem is given by the theorem of Hurwitz-Radon [10, 11, 16].

THEOREM I. *If* A_k^L *denotes the set of dimensions of possible* E_k-*modules then there exist integers* a_k^L *so that:*

1. $$A_k^L = \{ n\, a_k^L \} \qquad n = 1, 2, \ldots$$

One has:

2. $$a_{k+8}^L = 16\, a_k^L \qquad k > 1 .$$

3. *The first eight values of* a_k^L *are:* 1, 2, 4, 4, 8, 8, 8, 8.

Immediate corollaries are:

a) *The integer* a_k^L *is always a power of* 2.

b) E_k *occurs as an* E_k-*module* (i.e., $a_k^L = k$) *if and only if* $k = 1, 2, 4, 8$.

A lovely proof of this theorem is given by Eckmann in [8]. Very briefly, his argument takes this form.

Let G_k be the abstract group generated by the symbols 1, ε, $\sigma_1, \ldots, \sigma_k$, subject to the relations:

$$1 \text{ — identity}; \; \varepsilon^2 = 1; \; \varepsilon\sigma_i = \sigma_i \varepsilon \quad (i = 1, \ldots, k)$$
$$\sigma_i \sigma_j = \varepsilon\sigma_j \sigma_i \, , \; i \neq j \, ; \; \sigma_i^2 = \varepsilon \, , \; i = 1, \ldots, k \, .$$

Next let a G_k-module, W, be called special if ε acts as -1 on W. It is then easy to verify that;

E_n *admits the structure of an* E_k*-module, if and only if there exists a special* G_k*-module of dimension* n *over the real numbers.*

(In one direction this correspondence is obtained by sending σ_i into f_i and ε into -1, whenever $f_1, \ldots, f_k$ is an orthogonal k-system. This function is then seen to define a representation of G_k.)

We are thus led to seek the special G_k-modules and it will clearly suffice to find the irreducible ones among them. Eckmann determines these with the aid of the representation theory of finite groups. He first finds the irreducible special complex G_k-modules—it turns out that there is only one isomorphy class of these if k is odd, and that there are two such classes, however of the same dimension, when k is even—and then determines the real irreducible special G_k-modules by the Schur criterion: A complex G_k-module, W, is the complexification of a real one if and only if the character, χ_W, of W , satisfies the condition:

$$\sum \chi_w(g^2) > 0 \, , \quad g \in G_k \, .$$

It is at this point that the mod 8 dependance of the answer emerges.

So much, then, for the linear case. The theorem of Radon-Hurwitz of course also gives us information about problem I. Indeed if we denote by A_k the set of dimensions n, for which E_n admits an orthogonal k-system, then A_k contains A_k^L, so that A_k^L furnishes a lower bound for the set A_k.

Actually, at the present time there is no counter example to the conjecture that A_k equals A_k^L, however we are still far from a proof of such a fact. (Added in Proof: F. Adams has just established the validity of this conjecture.) The following theorem, due to I. M. James [12, 13, 14] possibly best describes the presently known information in the direction of this conjecture.

THEOREM II. *Let* A_k *denote the set of integers* n, *for which* E_n *admits an orthogonal* k-*system. Then there exist integers* a_k *with the property:*

Either $\quad A_k = \{ n\, a_k \} \qquad k = 1, 2, \ldots$

or $\quad A_k = \{ n\, a_k \} \qquad n = 2, 3, 4, \ldots$

further in the latter (exceptional) *case,* $k \leqq a_k \leqq 2k - 1$. *Finally, for all* k, $a_{k+1}/a_k = 1$ or 2.

The James theorem is clearly a great step towards the conjecture that $A_k = A_k^L$. The next step, one hopes, will be the elimination of the exceptional cases. (In this direction Adams has quite recently shown that in the exceptional case, a_k must actually equal $(2k - 1)$.)

We sketch the main lines of the proof briefly. Let $O_{n,k}$ be the Stiefel manifold of k-frames in E_n, and let $\pi\colon O_{n,k} \to O_{n,1}$ be the fiber-projection on the first element of this frame. Then an orthogonal k-system on E_n is equivalent to a section s: $O_{n,1} \to O_{n,k}$ of this fibering. (If $f_1, \ldots, f_k$ is an orthogonal k-system, s is defined by $s(x) = \{x, f_2(x), \ldots, f_k(x)\}$.) Now by the covering homotopy theorem the problem can be formulated entirely in terms of homotopy groups. Indeed, $\pi\colon O_{n,k} \to O_{n,1}$ admits a section if and only if $\pi_{n-1}(O_{n,k})$ maps onto $\pi_{n-1}(O_{n,1}) = Z$ under π_*. (Any element α, projecting onto the generator can be deformed into a section.) To recapitulate—from this point of view A_k consists of those integers n for which

$$\pi_*\colon \pi_{n-1}(O_{n,k}) \to \pi_{n-1}(O_{n,1})$$

is surjective.

The first step is now to show that if n and m are in A_k, then $n + m$ is again in A_k. In the linear case this is trivial enough—if E_n and E_m are E_k modules, then $E_n \oplus E_m$ is again an E_k module. The topological counter part to this argument is given by the join map λ of James, which takes $O_{n,k} * O_{m,k}$ into $O_{n+m,k}$. Here $*$ denotes the join, and λ is defined by:

If $x = \{x_i\}$ and $y = \{y_i\}$, $i = 1, ..., k$, are k-frames in E_n and E_m respectively, then $\lambda (x, t, y)$; $o \leqq t \leqq 1$; is the frame $\{x_i \cos \pi t/2 \oplus y_i \sin \pi t/2\}$ in $E_n \oplus E_m$.

In the usual manner the map

$$\lambda : O_{n,k} * O_{m,k} \to O_{n+m,k}$$

defines a pairing

$$\lambda_* : \pi_r (O_{n,k}) \oplus \pi_s (O_{m,k}) \to \pi_{r+s+1} (O_{n+m,k})$$

and the naturality conditions of λ_* relative to π_* easily yield the fact that A_k is closed under addition. To get further, one needs at least a partial subtraction law. The basic result in this direction is James's extension of the Freudenthal theorem:

GENERALIZED FREUDENTHAL THEOREM. *Suppose that* $n \in A_k$ *and let* $s \in \pi_{n-1} (O_{n,k})$ *project on a generator of* $\pi_{n-1} (O_{n,1})$. *Then* $s_* : \pi_i (O_{m,k}) \to \pi_{i+n} (O_{n+m,k})$ *defined by*: $s_* (y) = \lambda_* (s \otimes y)$, *is a bijection for* $i \leqq 2 (m - k + 1)$.

Roughly this theorem enables James to conclude that if $n + m \in A_k$ and n is small relative to $n + m$ then m is also in A_k. By subtracting the lowest integer in A_k successively as far as possible he then obtains theorem II.

James prove the generalized Freudenthal theorem by induction on k. For $k = 1$, we have precisely the Freudenthal theorem. The crucial fact here is a " boundary " formula of the type

$$\nabla \lambda_* (a \otimes b) = \lambda_* (\nabla a \otimes b) \pm a \otimes \lambda_* \nabla b$$

where ∇ is the boundary in the homotopy sequence of fiberings of the type $O_{n,k} \to O_{m,l}$. I will not describe it more precisely. However, this formula is the hardest and its proof the most geometric part of the whole theory.

Theorem II does not include all the presently known information about our vector-field problem. By means of cohomology operations one can, for instance find restrictions on the set A_k. I will not attempt to do justice to these but, rather, say a word about the parallelizability question which was settled two years ago.

The problem is: *For what* n, *does* S_{n-1} *admit an* (n — 1) *field, or put more geometrically, for what* n *can a global parallelism be defined on* S_{n-1} ? *In our notation the question is simply: When does* A_k *contain* k ?

The answer, due independently to Milnor [7] and Kervaire [15] asserts that, just as in the linear case, this phenomenon occurs only if $k = 1, 2, 4$ and 8.

At present several proofs of this result are known. The most topological proof is obtained by applying the work of Adams [1] on the decomposability of certain primary operations in terms of secondary ones. (This result becomes pertinent in view of the following construction. Corresponding to $\alpha \in \pi_{n-1}(O_{n,n})$ let ξ_α be the bundle determined over S_n, and let $X_\alpha = S_n \cup_\alpha e_{2n}$ be the complex obtained by forming the 1-point compactification of ξ_α. Then if α represents a section of $\pi\colon O_{n,n} \to O_{n,1}$ it is well known that $Sq^n\colon H^n(X_\alpha) \to H^{2n}(X_\alpha)$ is nontrivial. Now by Adams, this can occur only if $n = 1, 2, 4$ or 8.)

The original solutions of the parallelizability question were based on divisibility properties of the characteristic classes of vector bundles. Quite recently Atiyah and Hirzebruch brought another proof based on this principle, which is possibly the most satisfactory one. The main steps are:

If ξ is a (real) vector bundle over a complex X then $w(\xi)$—the Stiefel Whitney class—is a well determined element of $H^*(X; Z_2)$ which has component 1 in dimension O. This class is not affected by adding a trivial bundle to ξ. Now it is not hard to see that $k \in A_k$ is equivalent to the assertion: S_k *admits a vector bundle with* $w(\xi) \neq 1$. (If $\alpha \in \pi_{n-1}(O_{n,n})$ is a section, then $\pi_* \alpha$ (mod 2) can be identified with the component in dimension n of $w(\xi_\alpha)$ — ξ_α being the bundle over S_n determined by α.) With this as a starting point, the hard part of the parallelizability question is to show that $w(\xi) = 1$ whenever ξ is a vector bundle over a sphere of dimension > 8. Suppose then that $S_n = S_{8+m}$, $m \geqq 1$, so that $S_n = S_m \# S_8$ where $\#$ denotes the identification space obtained from $S_m \times S_8$ by collapsing the wedge $S_m \vee S_8$ in $S_m \times S_8$ to a point. Now the periodicity theorem for the stable orthogonal group asserts [5/6]:

Let X *be a finite* CW *complex, and let* ξ *be a* (real) *vector bundle over* $X \# S_8$. *Then there is a bundle,* ξ/λ, *over* X, *and a cannonical* 8 *dimensional bundle* λ *over* S_8, *so that* $\xi \equiv \xi/\lambda \otimes \lambda$, *where* $\otimes$ *denotes the reduced tensor product and the congruence is taken modulo trivial bundles.*

Concerning the reduced tensor product of two bundles ξ and η on X and Y one has to recall that $\xi \otimes \eta$ determines a bundle on $X \# Y$, and that $w(\xi \otimes \eta)$ is determined by ξ and η according to the law:

Let

$$w(\xi) = \prod_1^n (1+x_i), \quad n = \dim \xi ; \quad w(\eta) = \prod_1^m (1+y_j), \quad m = \dim \eta .$$

Then

$$w(\xi \oplus \eta) = \prod\nolimits_{i,j} (1+x_i+y_j)/\{ w(\xi) \}^m \cdot \{ w(\eta) \}^n$$

Now, if one takes a bundle ξ over S_{m+8}, $m > 1$; it follows from the periodicity formula that $w(\xi) = w(\xi/\lambda \otimes \lambda)$, and a purely algebraic estimate, using the fact that $w(\lambda) = 1 + u$, where u is the generator of $H^8(S_8; Z_2)$, and that λ has dimension 8, shows that $w(\xi) = 1$ under these conditions.

I would like to take up the " allied " problems next. We have been concerned with the question whether $\pi\colon O_{n,k} \to O_{n,1}$ has a section. Now this problem has an obvious analogue for the other two fields over the real numbers. The spaces $O_{p,q}$ are perfectly well defined over the complex numbers (unitary q-frames in Hermitian p-space) and also over the Quaternious (Symplectic q-frames in Symplectic p-space). Hence the question of whether $O_{n,k} \to O_{n,1}$ has a section is also meaningful over the complex numbers and the quaternions.

The method of James turns out to be applicable to these cases also. In fact, it yields the following stronger result:

THEOREM III. *Let* B_k, $[C_k]$ *denote the set of integers for which* $O_{n,k} \to O_{n,1}$ *has a section in the complex* [*quaternion*] *case. Then there exist positive integers* b_k $[c_k]$ *so that*

$$B_k = \{ n\, b_k \}$$
$$C_k = \{ n\, c_k \} .$$

In these two instances, then, there are no exceptional cases. The proof of James follows the earlier pattern. The result is stronger because the extended Freudenthal theorem is applicable in a greater range of dimensions. However, there is this great difference: In these two cases there is no known information about the linear case—in its strongest formulation there are no linear orthogonal k-systems for $k > 1$—in particular it is not apriori clear that B_k is nonempty for $k > 1$. To show that indeed $O_{n,k} \to O_{n,1}$ has a section for some n, James again uses his boundary formula essentially to derive this fact from the finiteness of the stable homotopy groups of the spheres.

There is another path to this theorem. Motivated by certain other results of James, this approach has recently been perfected by M. Atiyah [2]. It proves that B_k consists of multiples of a certain integers b_k by essentially identifying B_k with the kernel of a homomorphism of a cyclic group. I will discuss only the complex case, as the quaternion case is entirely similar.

Several steps are involved. First we reformulate our problem once again. As our concern is now with the behavior of π_{2n-1} under the projection $O_{n,k} \to O_{n,1}$ we may replace $O_{n,k}$ by its $2n$-skeleton, which can be constructed in a very simple manner when n/k is large. Let P_n denote the projective space of the one-dimensional subspaces of complex n-space C_n. (Thus P_n is the projective space of complex dimension $n - 1$.) Next, define $P_{n,k}$ to be the identification space P_n/P_{n-k}, the inclusion $P_{n-k} \subset P_n$ being induced by the inclusion $C_{n-k} \subset C_n$. We clearly have a natural projection π': $P_{n,k} \to P_{n,1}$, and it can be shown that the suspension of π', that is $E \circ \pi'$: $EP_{n,k} \to EP_{n,1} = S_{2n-1}$, represents the projection $O_{n,k} \to O_{n,1}$ in the pertinent dimension—at least if n/k is large. Using cohomology operations to eliminate low ratios of this integer, one concludes, that $O_{n,k} \to O_{n,1}$ has a section, if and only if $(E' \circ \pi)_*$ is surjective in dimension $2n - 1$. Finally, again using the apriori estimate on n/k one can redefine B_k in terms of purely " stable " notions.

The integer n $\in B_k$ *if and only if some suspension of the map* π': $P_{n,k} \to S_{2n-1}$, *admits a right homotopy inverse.*

Precisely then, the condition is that there should exist an m, and a map f: $E^m S_{2n-1} \to E^m P_{n,k}$ so that $E^m \pi' \circ f$ be homotopic to 1. Colloquially one may also put it this way: "The top cycle of $P_{n,k}$ should become spherical after a suitable number of suspensions." James calls this condition S-reducibility.

To proceed further we need the notion of the generalized J-homomorphism, and of the twisted suspension.

Let X be a finite CW complex. We write $\widetilde{KO}(X)$ for the suspension classes of real vector bundles over X (see [3, 9]). Thus two bundles ξ and η determine the same element, $[\xi] = [\eta]$, in $\widetilde{KO}(X)$ if after suitable trivial bundles are added to both they become isomorphic. Next define $J(X)$ as the set of equivalence classes of vector-bundles over X, in which two bundles are considered equal if after suitable trivial bundles are added to them, their unit sphere-bundles are of the same fiber-homotopy type. Finally J shall denote the projection $\widetilde{KO}(X) \to J(X)$. The Whitney sum now defines a group structure in both these sets and makes J into a homomorphism. So interpretted J is the generalized J-homomorphism. A first observation is now,

PROPOSITION 1. J (X) *is a finite group.*

The proof follows more or less directly from finiteness of the stable homotopy of the spheres and the definition of fiber homotopy type.

Finally the twisted suspension of X, by a vector bundle ξ (over X) is defined as the one-point compactification of ξ, and will be denoted by X^{ξ}. The terminology is justified by this formula:

$$X^{(\xi+1)} = E \cdot X^{\xi}$$

where 1 stands for the trivial bundle and E denotes suspension as before. One also needs the convention that when ξ has dimension O, then X^{ξ} is to be the disjoint union of X and a point.

This construction is pertinent to our discussion for the following reason: Let P_k be the orthogonal projective space to P_{n-k} in P_n, and let τ denote its normal bundle. Then it is geometrically clear that τ can be identified with $P_n - P_{n-k}$. Hence $P_{n,k} = P_k^{\tau}$. The bundle τ splits into the direct sum of (complex) line-bundles as is also evident because P_k is a complete intersection in P_n.

Thus, if ξ is the normal bundle of in P_k P_{k+1} then $P_{n,k}$ is given by:

$$P_{n,k} = P_k^{(n-k)\xi}$$

To return to the general situation—First there is the following rather easy relation between J and the twisted suspension:

PROPOSITION 2. *Let ξ be a vector bundle over* X. *Then* X^ξ *is of the same* S-*type as* X *if and only if* $J[\xi] = 0$.

(Here, as usual, two spaces are of the same S-type if suitable suspensions of them are of the same homotopy type.)

Suppose now that M is a compact differentiable manifold. If ξ is a vector-bundle over M, the S-reducibility of M^ξ is defined exactly as it was for $P_{n,k}$—the top cycle of M^ξ has to be stably spherical. Let ν be the normal bundle of M imbedded in some high dimensional sphere S_N. By collapsing the exterior of a tubular neighborhood of M in S_N we obtain a map $S_N \to M^\nu$ which clearly establishes M^ν as S-reducible. This argument makes the following proposition plausible:

M^ξ *is* S-*reducible if and only if* $J([\xi] - [\nu]) = 0$.

By replacing M with P_k this last formula now immediately yields the theorem of James. Indeed, in this case $[\nu] = -k[\xi]$ as is well known. Thus $P_{n,k} = P_k^{(n-k)\xi}$ is S-reducible if and only if $J(n[\xi]) = 0$. Because $J(P_k)$ is finite, and J is a homomorphism the theorem follows.

The last formula is really a special case of the following *duality theorem* of Atiyah, which was also independently proved by A. Shapiro and the Author.

DUALITY THEOREM. *Let* X *be a differentiable manifold, and let* ξ *and* ξ' *be two vector bundles over* X *so that* $[\xi'] = -([\xi] + [\tau])$, *where* τ *is the tangent bundle of* X. *Then the* S-*types of* X^ξ *and* $X^{\xi'}$ *are dual to each other in the sense of Spanier Whitehead* [17]:

$$D[X^\xi] = [X^{-\xi-\tau}]\,.$$

A remark is now in order as to why the real case cannot be treated in this manner. Actually one can procede quite similarly at first. In the real case $O_{n,k}$ is approximated by the real analogue of $P_{n,k}$, rather than by $EP_{n,k}$ as it was in the present situation, however this is no serious drawback. The exceptional case can occur precisely because one is not always able to eliminate low ratios of n/k apriori, so that the S-reducibility of $P_{n,k}$ is not necessarily an equivalent problem to the existence of a section to the fibering $O_{n,k} \to O_{n,1}$. The S-reducibility of the real $P_{n,k}$ of course has the same sort of answer as in the complex case.

In conclusion let me report on estimates which Atiyah and Todd obtained for the b_k of theorem II [3]. Let $\lambda_p(N)$ denote the power to which the prime p occurs in the integer N. Now let integers M_k be defined by the formula:

$$\lambda_p(M_k) = \begin{cases} \sup\,(r + \lambda_p(r))\,, \quad 1 \leqq r \left[\dfrac{k-1}{p-1}\right], & \text{if } p \leqq k \\ 0 \qquad \text{if } p > k\,. \end{cases}$$

THEOREM III. *The integers* b_k *of theorem II* (for the complex case) *are divisible by* M_k.

The principle on which this estimate is based is the following one. As we have seen, our question is really: For what values of n is the top cycle of $P_{n,k}$ " stably spherical ". That is, when does this homology class become spherical after a suitable number of suspensions.

In short, we need criteria for stably spherical homology classes. The following simple procedure clearly yields such criteria. Suppose B is a space in which the stably spherical classes are already known. Then if $u \in H_i(X)$ is a homology class in X, it will be stably spherical only if for every map f: $X \to B$, $f_* u$ is stably spherical in $H_*(B)$. Such a criterion is of course only effectively applicable if we know how to compute (1) the set of homotopy classes of maps of X into B and (2) the homorphisms these classes induce in cohomology.

The best known application occurs when B is an Eilenberg-Maclane space. Here there are no stable spherical classes except the lowest dimensional ones—(in the stable range). In this way one obtains the criterion that u is stably spherical only if the value of any stable primary cohomology operation on a lower dimensional class, vanishes on u. (This criterion can be applied to our problem, however it yields considerably weaker results than those given by theorem III.)

The results of Atiyah Todd, are in fact obtained by using for their testing space B, the universal base space, B_U, of the infinite unitary group. As a consequence of the periodicity [5]: $\Omega^2(Z \times B_U) = Z \times B_U$ one can determine the spherical classes in B_U rather easily, and for stable spherical classes one can derive this criterion: *There is a rational cohomology class* ch (with components in all dimensions) *in* $\mathrm{H}^*(\mathrm{B}_U; \mathrm{Q})$, *such that if* u *is a stably spherical class in* $\mathrm{H}_k(\mathrm{B}_U)$ *then* ch (u) *must be an integer.*

Thus, if we write $\widetilde{KU}(X)$ for the homotopy classes of maps of X into B_U, and for $\xi \in \widetilde{KU}(X)$ write $ch(\xi) = \xi_* \,.\, ch$, then we have the criterion:

$\mathrm{u} \in \mathrm{H}_k(\mathrm{X})$ *is stably spherical only if for each* $\xi \in \widetilde{\mathrm{KU}}(\mathrm{X})$, $\mathrm{ch}(\xi) \,.\, \mathrm{u}$ *is an integer.*

It is this criterion which yields the Atiyah Todd theorem modulo some rather delicate number theory.

How does one carry out the steps (1) and (2) of our program in this case ? Here it is only fair to admit, that the space B_U is not an ad-hoc testing space, but rather that it is more or less God given. Indeed, by virtue of the classifying theorems for bundles, $\widetilde{KU}(X)$ can be interpretted geometrically as the suspension classes of complex vector bundles over X. Further, if $\xi \in \widetilde{KU}(X)$ then the element $ch(\xi)$ in $H^*(X; Q)$ is a particular characteristic class of ξ about which a lot is known. For instance if ξ is the normal bundle of P_n in P_{n+1}, then $ch(\xi) = e^x - 1$ where $x \in H^2(P_n; Z)$ is a generator. Now, using the known functorial properties of ch, it follows that $ch(\xi^k) = (e^x - 1)^k$, where ξ^k is $\xi \otimes \dots \otimes \xi$ (k times) in the sense of the reduced tensorproduct. Thus, if we restrict ξ^m with $m \geqq n - k$ to P_{n-k} we get an element of $\widetilde{KU}(P_{n-k})$ with vanishing character.

It is a theorem that if X is a torsion free finite CW complex then $ch\colon \widetilde{KU}(X) \to H^*(X; Q)$ is injective. This was first proved by F. Peterson—directly by obstruction theory from the evaluation of $\pi_i(Z \times B_U)$ as Z if i is even and 0 when i is odd. A proof which possibly goes more to the heart of the matter emerges from the point of view of Atiyah and Hirzebruch. They define the groups:

$$KU^i(X) = \pi[E^{-i}X;\ Z \times B_U] \qquad i \leqq 0$$

where $\pi[A, B]$ denotes homotopy classes of maps. In this terminology the periodicity formula: $\Omega^2(Z \times B_U) = Z \times B_U$ is expressed by:

$$KU^i(X) = KU^{i+2}(X) \qquad i \leqq -2\,.$$

Now Atiyah and Hirzebruch use this recurrence to define $KU^i(X)$ for all integers i, and then show that the resulting functor $X \to \{KU^i(X)\}$ satisfies all the axioms of a cohomology theory —except for the dimension axiom. Further they observe that the uniqueness theorem of Eilenberg-Steenrod can be generalized to yield a spectral sequence relating $E_2 = H^*(X;\ KU^*(p))$ to $KU^*(X)$. (Here $KU^i(p)$—the KU—theory of a point—is Z if i is even and 0 otherwise.) This sequence immediately implies the proposition. (See [8].)

To return to our bundles ξ^m on P_n. By the proposition just discussed the restriction of ξ^m to P_{n-k} will be trivial if $m \geqq n - k$. By trivializing this element on P_{n-k} we obtain bundles ξ^m on $P_{n,k}$ which under the projection π: $P_n \to P_{n,k}$ go over into ξ^m, $m \geqq n - k$. In particular, $\pi^*\, ch\,(\xi^m) = (e^x - 1)^m$. Thus in any case we obtain these criteria the S-reducibility of $P_{n,k}$:

$P_{n,k}$ is S-*reducible only if the coefficient of* x^{n-1} *in* $(e^x - 1)^m$, $m \geqq n - k$, *is an integer.*

This is the number theoretical condition from which Atiyah and Todd deduce theorem III. Their result is the best possible one obtainable from the test-space B_U, because one can show quite easily, with the spectral sequence alluded to earlier, that the elements 1, ξ^m; $n - 1 \geqq m \geqq n - k$; form a base of $KU\,(P_{n,k})$.

BIBLIOGRAPHY

[1] Adams, J., On the nonexistence of elements of Hopf invariant one. *Ann. of Math.*, 72 (1960), 20-104.
[2] Atiyah, M., *Thom Complexes* (to be published).

[3] Atiyah, M., and F. Hirzebruch, Riemann-Roch theorems for differentiable manifolds. *Bull. Amer. Math. Soc.*, 65 (1959), 276-281.

[4] —— and J. Todd, On complex Stiefel Manifolds. *Proc. Camb. Phil.* Soc. (1960).

[5] Bott, R., The stable homotopy of the classical groups. *Ann. of Math.*, 70 (1959).

[6] —— Quelques remarques sur les théorèmes de périodicité. *Bull. Soc. Math. France* (1959), 293-310.

[7] —— and J. Milnor, On the parallelizability of the spheres. *Bull. Amer. Math. Soc.*, 64 (1958), 87-89.

[8] Eckmann, B., Gruppentheoretischer Beweis des Satzes von Hurwitz-Radon über die Komposition quadratischer Formen. *Comm. Math. Helv.*, 15 (1943), 358-366.

[9] Hirzebruch, F., A Riemann-Roch theorem for differentiable manifolds. *Séminaire Bourbaki* (1959), n° 177.

[10] Hurwitz, A., Ueber die Komposition des quadratischen Formen. *Math. Ann.*, 88 (1923), 1-25.

[11] —— Ueber die Komposition quadratischer Formen von beliebig vielen Variabeln. *Nachr. Ges. d. Wiss. Göttingen* (1898), 309-316.

[12] James, I., The intrinsic join. *Proc. London Math. Soc.* (3), 8 (1958), 507-35.

[13] —— Cross sections of Stiefel manifolds. *Ibid.*, 536-47.

[14] —— Spaces associated with Stiefel Manifolds. *Ibid.* (3), 9 (1959), 115-40.

[15] Kervaire, M., Nonparallelizability of the n-sphere for $n > 7$. *Proc. Nat. Acad. Sci. U.S.A.* (1958), 283-283.

[16] Radon, J., Lineare Scharen orthogonaler Matrizen. *Abh. Sem. Hamburg*, I (1923), 1-14.

[17] Spanier, E. and J. H. C. Whitehead, Duality in homotopy theory. *Mathematica*, 2 (1955), 56-80.

Department of Mathematics
Harward University
Cambridge, Mass. U.S.A.