4. The algebras of reduced power operations.

and the Künneth formula) and the single new operation Ψ. This simplifies the derivation of the properties of the Sq^i, and illuminates their origin.

Note that the projection $W \times_\pi K^2 \to P$ is a fibration with fibre K^2. For each $x \in H^q(K)$, $x \otimes x$ is a cohomology class of the fibre. The element Ψx is a canonical extension of $x \otimes x$ to a class on the total space.

4. The algebras of reduced power operations.

The definition of the reduced powers, given above for complexes, extends to the Čech cohomology of general spaces by taking direct limits of the operations in the nerves of coverings. The extension to the singular theory, by the method of acyclic models, has been carried through by Araki [4].

The main property of the squares is that

$$\mathrm{Sq}^i\colon H^q(X; Z_2) \to H^{q+i}(X; Z_2)$$

is a homomorphism for each space X and each $i \geqq 0$, and if $f\colon X \to Y$ is a mapping, Sq^i commutes with the induced homomorphism f^* of cohomology. The principal algebraic properties are

(4.1) $$\mathrm{Sq}^0 = \text{identity}.$$

(4.2) $\mathrm{Sq}^1 =$ the Bockstein operator β of the coefficient sequence

$$0 \to Z_2 \to Z_4 \to Z_2 \to 0\,.$$

(4.3) If $\dim x = n$, then $\mathrm{Sq}^n x = x^2$.

(4.4) If $\dim x = n$, then $\mathrm{Sq}^i x = 0$ for all $i > n$.

(4.5) The Adem relations [2]: If $a < 2b$, then

$$\mathrm{Sq}^a\,\mathrm{Sq}^b = \binom{b-1}{a}\mathrm{Sq}^{a+b} + \sum_{j=1}^{[a/2]}\binom{b-j-1}{a-2j}\mathrm{Sq}^{a+b-j}\,\mathrm{Sq}^j\,.$$

(4.6) The Cartan formula [6]: If $x, y \in H^*(X;\, Z_2)$, then

$$\mathrm{Sq}^i(xy) = \sum_{j=0}^{i}(\mathrm{Sq}^j x)(\mathrm{Sq}^{i-j} y)\,.$$

The generalization of the reduced powers to primes $p > 2$ takes on a somewhat unexpected form. Many of the terms in the formula corresponding to 3.3 prove to be zero. The remaining terms can be expressed using a sequence of homomorphisms

$$\mathscr{P}^i_p \colon H^q(X; Z_p) \to H^{q+2i(p-1)}(X; Z_p)\,, \qquad i = 0, 1, 2, \ldots$$

and the Bockstein operator β of the coefficient sequence

$$0 \to Z_p \to Z_{p^2} \to Z_p \to 0\,.$$

(The analogy with $p = 2$ associates $\mathscr{P}^i_2$ with Sq^{2i}.) Their algebraic properties are

(4.7) $\mathscr{P}^0 =$ identity.

(4.8) If $\dim x = 2n$, then $\mathscr{P}^n x = x^p$.

(4.9) If $2i > \dim x$, then $\mathscr{P}^i x = 0$.

(4.10) The Adem-Cartan relations [3, 9]: If $a < pb$, then

$$\mathscr{P}^a \mathscr{P}^b = \sum_{i=0}^{[a/p]} (-1)^{a+i} \binom{(p-1)(b-i)-1}{a-ip} \mathscr{P}^{a+b-i} \mathscr{P}^i\,.$$

If $a < pb + 1$, then

$$\mathscr{P}^a \beta \mathscr{P}^b = \sum_{i=0}^{[a/p]} (-1)^{a+i} \binom{(p-1)(b-i)}{a-ip} \beta \mathscr{P}^{a+b-i} \mathscr{P}^i + \sum_{i=0}^{[a-1/p]} (-1)^{a-1+i} \binom{(p-1)(b-i)-1}{a-ip-1} \mathscr{P}^{a+b-i} \beta \mathscr{P}^i\,.$$

(4.11) The Cartan formula [18]: If $x, y \in H^*(X; Z_p)$, then

$$\mathscr{P}^i(xy) = \sum_{j=0}^{i} (\mathscr{P}^j x)(\mathscr{P}^{i-j} y)\,, \qquad \beta(xy) = (\beta x)y + (-1)^{\dim x} x(\beta y)\,.$$

The algebra $\mathscr{A}_2$ of the squaring operations is defined to be the graded associative algebra over Z_2 generated by the Sq^i ($i = 0, 1, 2, \ldots$) subject to the relations 4.1 and 4.5. Similarly, for $p > 2$, $\mathscr{A}_p$ is the graded associative algebra over Z_p generated by β and the $\mathscr{P}^i$ subject to the relations 4.7, 4.10 and $\beta^2 = 0$. Degrees are defined by $\deg(Sq^i) = i$, $\deg(\beta) = 1$, $\deg(\mathscr{P}^i)$

$= 2i(p-1)$; and the degree of a monomial in the generators is the sum of the degrees of the factors. After these definitions, it follows readily that, for each p, the cohomology $H^*(X; Z_p)$ of a space X is a graded $\mathscr{A}_p$-module.

As an abstract algebra, $\mathscr{A}_p$ has a complicated structure. It is, of course, non-commutative. The Adem-Cartan relations give a kind of commutation law. A monomial in the generators

$$\beta^{\varepsilon_0} \mathscr{P}^{r_1} \beta^{\varepsilon_1} \mathscr{P}^{r_2} \ldots \mathscr{P}^{r_k} \beta^{\varepsilon_k} \qquad (\varepsilon_j = 0 \quad \text{or} \quad 1)$$

is called *admissible* if $r_j \geqq pr_{j+1} + \varepsilon_j$ for $j = 1, 2, \ldots, k-1$ and $r_k \geqq 1$. The Adem-Cartan relations are rules for expressing inadmissible monomials in terms of admissible ones. Cartan has shown [9] that the admissible monomials form a vector space basis for $\mathscr{A}_p$. Thus there is a *normal form* for an element of $\mathscr{A}_p$.

Another consequence of the relations is the following result of Adem [3]:

4.12. *The algebra $\mathscr{A}_p$ is generated by β and the $\mathscr{P}^{p^i}$ for* $i = 0, 1, 2, \ldots$; *and $\mathscr{A}_2$ is generated by the* Sq^{2^i} *for* $i = 0, 1, 2, \ldots$.

Let us see how this is proved for $\mathscr{A}_2$. Assume, inductively, that, for $j < n$, each Sq^j is in the subalgebra generated by the Sq^{2^i}. If n is not a power of 2, then $n = a + 2^k$ where $0 < a < 2^k$. Set $b = 2^k$ and apply 4.5. The coefficient in 4.5 of $\mathrm{Sq}^{a+b} = \mathrm{Sq}^n$ is congruent to 1 mod 2. It follows that Sq^n is decomposable as a sum of products of Sq^j with $j < n$. The inductive hypothesis now implies that Sq^n is in the subalgebra of the Sq^{2^i}.

5. Non-realizability as cohomology algebras.

The preceding results will now be used to show that many of the graded algebras $F(R, n)^h$ on one generator of dimension n and height h are not realizable. Recall that $F(R, n)^2$ is realized by the n-sphere for each n and any ring R. So we shall restrict attention to the cases $2 < h \leqq \infty$.

First let $R = Z_2$, and assume that $F(Z_2, n)^h$ is realized by a space X. Let $x \in H^n(X; Z_2)$ be the generator of $H^*(X; Z_2)$. Since $h > 2$, x^2 is not zero. By 4.3, $\mathrm{Sq}^n x = x^2$ is not zero.