HOLOMORPHIC MAPPINGS OF COMPLEX MANIFOLDS

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1. A complex manifold is, briefly speaking, a connected manifold with local complex coordinates defined up to a holomorphic transformation. Examples of complex manifolds include the number space C_m and the projective space P_m of dimensions m. For $m = 1$ these are known in function theory as the Gaussian plane and the Riemann sphere respectively.

A holomorphic mapping of a complex manifold M of dimension m into another one N of dimension n is a continuous mapping f such that, if $\zeta_1, \ldots, \zeta_m$ are the local coordinates at a point $\zeta \in M$ and $z_1, \ldots, z_n$ are local coordinates at the image point $f(\zeta) \in N$, the mapping is locally defined by the equations

$$z_i = z_i(\zeta_1, \ldots, \zeta_m) , \qquad l \leqq i \leqq n , \tag{1}$$

where the functions at the right-hand side are holomorphic functions in their arguments. By this definition, a holomorphic mapping $f: C_1 \rightarrow P_1$ is precisely a meromorphic function in classical function theory.

The first question that arises is the question of existence. For the condition of a holomorphic mapping is so strong that it is not clear that, for given complex manifolds M, N, a holomorphic mapping $f: M \rightarrow N$ should exist which is not a constant (i.e., one that the image $f(M)$ is not a single point of N). In fact, if M, N are compact Riemann surfaces (a Riemann surface is a complex manifold of dimension one), then a non-constant holomorphic mapping $f: M \rightarrow N$ exists only when $g(M) \geqq g(N)$, where $g(M)$, $g(N)$ are the genera of M, N respectively. This well-known result can be derived as a consequence of the Riemann-Hurwitz formula (cf. § 2).

A more elementary fact is the result that every holomorphic function on a compact complex manifold is a constant. From this it follows that every holomorphic mapping of P_m into a complex torus of dimension n is a constant, because P_m is simply connected and the complex torus has C_n as its universal covering space.

The above result can be generalized. We recall that an analytic set on a complex manifold M is a set E satisfying the condition that, if $\zeta_0 \in E$, there exist s holomorphic functions $f_1, \ldots, f_s$ in a neighborhood of ζ_0 such that the intersection of E with the neighborhood is defined by the equations $f_1 = \ldots = f_s = 0$. Then the following theorem is known [6, p. 356]: Let $f: M \to N$ be a holomorphic mapping such that M is compact and that every compact analytic set of N consists of a finite number of points. Then f is constant.

2. While these results are of interest, it seems desirable to formulate some problems of general scope on holomorphic mappings. I would consider the following a fundamental one: Given a holomorphic mapping $f: M \to N$. To determine relations between the invariants of the manifolds M, N and the invariants which arise from the mapping f.

A first illustration of this problem is the Riemann-Hurwitz formula on the holomorphic mapping $f: M \to N$ of compact Riemann surfaces. The formula can be written

$$2 - 2g(M) + w = d(2 - 2g(N)) , \tag{2}$$

where d is the degree of the mapping and w is the index of ramification, i.e., the sum of the orders of the points of ramification. The genera $g(M)$, $g(N)$ are invariants of M, N themselves, while d, w depend on the mapping.

Another set of relations of this nature consists of the Plücker formulas for an algebraic curve. Let an algebraic curve be defined by a holomorphic mapping $f: M \to P_n$, where M is a compact Riemann surface. Suppose that the curve is non-degenerate, i.e., that the image $f(M)$ does not belong to a sub-space of dimension $\leqq n - 1$. To this curve is defined the *pth* associated curve $f^p: M \to G(n, p)$, $0 \leqq p \leqq n - 1$, formed by the osculating projective spaces of dimension p, where $G(n, p)$ is the Grassmann manifold of all p-dimensional projective spaces in P_n $(G(n, 0) = P_n)$. $f^p(M)$ defines a cycle in $G(n, p)$, which is homologous to a positive integral multiple ν_p of the fundamental two-cycle of $G(n, p)$. The integer $\nu_p \geqq 0$ is called the order of rank p of our algebraic curve. Geometrically it is the number of points of the curve at which the osculating spaces of dimension p meet a fixed generic linear space of dimension $n-p-1$ of P_n. A stationary point of order p is one at which the *pth* associated curve has a tangent with a contact of higher order. The stationary points are isolated and a positive index can be associated to each of them. Let $w_p \geqq 0$ be the sum of indices at the stationary points of rank p. Then Plücker's formulas are

$$(3) \qquad -w_p-\nu_{p-1}+2\nu_p-\nu_{p+1} = 2-2g(M)\,, \qquad 0\leqq p\leqq n-1\,.$$

Here the right-hand side is an invariant of M, while the left-hand side involves quantities which depend on the mapping.

For non-singular algebraic varieties a much more profound relation between invariants of manifolds and quantities depending on a holomorphic mapping is given by Grothendieck's Riemann-Roch theorem [1]. We will not dwell on a discussion of this theorem. It suffices to say that the theorem contains as a special case the Riemann-Hurwitz formula. Applying the theorem of Grothendieck and the classification of singularities by Thom, I. R. Porteous [5] derived relations between the characteristic classes of non-singular algebraic varieties under the following simple types of mappings: *a)* dilatations; *b)* ramified coverings with singularities of a relatively simple type.

It will be natural to expect that the relations answering our fundamental problem have a bearing on the existence problem of holomorphic mappings. An example is the non-existence theorem of holomorphic mappings between compact Riemann surfaces in § 1 derived as a consequence of the Riemann-Hurwitz formula. But our fundamental problem seems to be wider in scope.

A natural counter-part of the existence problem is the uniqueness problem, namely the determination of a holomorphic mapping by its restriction to a certain subset of the original manifold. Very little seems to be known along this line. As an example I wish to state the following so-called Riemann's theorem [6, p. 343]: Let f: $M \to N$ be a continuous mapping of a complex manifold M into another N. Let E be a nowhere dense analytic set in M. If the restriction of f to $M - E$ is holomorphic, the same is true for f itself.

3. Another important problem on holomorphic mappings is the study of the properties of the image set. If f: $M \to N$ is a holomorphic mapping and M is compact, then $f(M)$ is an analytic set. If $N = P_n$, then a famous theorem of Chow says that $f(M)$ is an algebraic set. (We recall that a subset $E \subset P_n$ is called algebraic, if there exist q polynomials $g_1, \ldots, g_q$ in the $n+1$ homogeneous coordinates of P_n such that E is defined by the equations $g_1 = \ldots = g_q = 0$.)

The case that M, N are of the same dimension has particular properties for the following reasons: 1) M, N are oriented manifolds and f preserves orientation; 2) it will be possible to compare the local degree of the mapping with the global degree. The results so obtained are valid for more general mappings. In fact, the following theorem was proved by S. Sternberg and R. G. Swan [9]: Let M, N be two oriented n-dimensional differentiable manifolds, with M compact and N connected. Let f: $M \to N$ be a differentiable mapping, whose Jacobian $J(f)$ is non-negative. Then either $J(f) \equiv 0$ or N is also compact, f is onto, and f has a positive degree on each component of M on which $J(f) \not\equiv 0$.

In particular, suppose M be connected and compact, and $J(f) \not\equiv 0$. Then N is compact, the degree $d(f)$ of the mapping is positive, and every point $a \in N$ is covered $d(f)$ times when counted with the proper multiplicity. Since N is compact, we can equip it with a riemannian metric, so that the total volume of N is 1. Let $v(M)$ be the volume of the image of M under f, and let $n(a)$ be the local degree of f at a, i.e., the number of times that a is covered by $f(M)$. Then we have

$$d(f) = n(a) = v(M) \,. \tag{4}$$

These results should be considered as a starting-point of the theory of value distributions in complex function theory, the essential difference being that, in the latter case, M is non-compact.

4. The study of mappings $f: M \to N$ where M is non-compact is radically different from the compact case and much finer analytical considerations will be necessary. The natural idea is to exhaust M by a family of compact domains with boundary, D_t, as $t \to \infty$, and to study the restriction of f to D_t. The asymptotic behavior of the geometrical quantities introduced for the restricted mappings $f \mid D_t$ as $t \to \infty$ will then be the main concern of the problem.

The problem which generalizes (4) to the case of a domain with boundary can be stated as follows: Let M and N be two connected, oriented n-dimensional C^∞-manifolds with M non-compact and N compact. Let f: $M \to N$ be a C^∞-differentiable mapping, whose Jacobian $J(f)$ is $\geqq 0$ and $\not\equiv 0$. Let $a \in N$, and let $f \mid D$ be the restriction of f to a compact domain $D \subset M$, such that the image of the boundary ∂D of D does not contain a. Equip N with a riemannian metric with total volume 1, and denote by $v(D)$ the volume of $f(D) \subset N$. Let $n(a, D)$ be the number of times that the point a is covered by $f(D)$. Our problem is to express the difference $n(a, D) - v(D)$ as an integral over ∂D.

An explicit formula solving this problem, which will then be a generalization of (4), is called the first main theorem. A most convenient way to derive such a formula is by applying the theory of harmonic differential forms on a compact riemannian manifold [7] and proceeds as follows:

We consider the manifold N and denote by Φ its volume element. Let δ_a be the Dirac measure with singularity at a. Then Φ and δ_a are both currents of dimension zero and their difference $\Phi - \delta_a$ is orthogonal to the harmonic form Φ. It follows from the fundamental existence theorem on harmonic integrals on a compact riemannian manifold that the equation

$$\Delta S = \delta_a - \Phi \tag{5}$$

where S is a current of dimension zero and Δ is the Laplacian, has a solution in S and that S is a differential form of degree

n in $N - a$. Moreover, the solution S is defined up to an additive harmonic form. Put

$$(6) \qquad \Lambda = \delta S \,,$$

where δ is the codifferential. Then, under the above hypotheses, we have the " first main theorem ":

$$(7) \qquad n(a, D) - v(D) = \int_{f(\partial D)} \Lambda \,.$$

In order to derive geometrical consequences from (7), it will be necessary to study the integral at its right-hand side, particularly its asymptotic behavior. Formula (7) contains as a special case the classical first main theorem in the theory of value distributions of meromorphic functions, but is of course much more general in scope. One can say that the reason which accounts more than any other for the properties of value distributions of meromorphic functions is the remarkable behavior of the boundary integral in (7).

I have carried out the study of the boundary integral in (7) for the case that $M = C_n$, $N = P_n$. Let $\zeta_1, \ldots, \zeta_n$ be the coordinates in C_n, and let D_t be the ball defined by

$$(8) \qquad \zeta_1 \bar{\zeta}_1 + \ldots + \zeta_n \bar{\zeta}_n \leqq t^2 \,.$$

Let

$$(9) \qquad \Omega_0 = \frac{i}{2} (d\zeta_1 \wedge d\bar{\zeta}_1 + \ldots + d\zeta_n \wedge d\bar{\zeta}_n) \,,$$

and let Ω be the fundamental two-form of the elliptic Hermitian metric in P_n such that $\int_{P_n} \Omega^n = 1$. We put

$$(10) \qquad v_k(t) = \int_{D_t} f^* \, \Omega^{n-k} \wedge \Omega_0^k \,, \quad 0 \leqq k \leqq n \,,$$

so that $v_0(t)$ is the volume of $f(D_t)$. By estimating the boundary integral in (7) and applying integral-geometric considerations, the following geometrical result is derived [3]:

Let $f \colon C_n \to P_n$ be a holomorphic mapping which satisfies the following conditions: 1) The function $T(t) = \int_{t_0}^{t} \frac{v_0(\tau)\, d\tau}{\tau^{2n-1}} \to \infty$;

2) $\int_{t_0}^{t} (v_1' (\tau)\, d\tau)/\tau^{2n} = o\,(T\,(t))$. Then the set $P_n - f\,(C_n)$ is of measure zero.

It is well-known that for an arbitrary holomorphic mapping $f\colon C_n \to P_n$, the set $P_n - f\,(C_n)$ may contain some open subset of P_n, so that the conclusion will certainly not be true without some supplementary condition. On the other hand, it is not necessary to suppose the holomorphy of f, for even in the classical case of value distributions the main results are true for quasi-meromorphic functions. It would be an interesting problem to find the proper restrictions on f for the above conclusion to be true.

5. The fundamental problem posed in the beginning of § 2 has a meaning also for the case of a holomorphic mapping $f\colon M \to N$, where M, N are compact complex manifolds, M being now with boundary. If both M and N are Riemann surfaces, the result so obtained forms a generalization of the Riemann-Hurwitz formula. Such a result is easily derived as a consequence of the Gauss-Bonnet formula. In the particular case when $N = P_1$, this is called the second main theorem of the theory of value distributions of meromorphic functions and constitutes the core of the theory.

By simply writing down the generalized Riemann-Hurwitz formula, one can derive in a purely differential-geometric way the following theorem *): Let $f\colon D \to N$ be a holomorphic mapping, where D is the pointed disk $0 < |\zeta| < 1$ and N is a compact Riemann surface of genus > 1. Then f can be extended as a holomorphic mapping of the whole disk $|\zeta| < 1$ into N.

Similarly, by a combination of the first and second main theorems, one can generalize the defect relations on meromorphic functions to holomorphic mappings $f\colon M \to P_1$, where M is a non-compact Riemann surface such that it can be compactified, as a Riemann surface, by the addition of a finite number of points. In the case that the image Riemann surface N is a complex torus, one derives in this way the result that the defect at every point $a \in N$ is zero. Geometrically the latter means that N is " evenly " covered by the image of M.

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*) I am indebted to J.-P. Serre for pointing out this conclusion to me.

All these seem to justify the emphasis we have put on our fundamental problem. Unfortunately, for higher dimensions, even when the image manifold is P_n, our knowledge on the problem is still very limited. For a holomorphic mapping $f: M \to P_n$, with M compact, this leads us back to the old theory of projective invariants in algebraic geometry. With recent advances in algebraic geometry, it might be possible and worthwhile to organize the classical results in a better form. The case of non-compact M awaits much further work.

I hope to have pointed out a few guiding ideas on the subject of holomorphic mappings. Only the future can tell whether the topic will lead to results of general mathematical interest. I cannot help to feel, however, that so long as the complex structure remains a subject of investigation, the study of holomorphic mappings should be a logical objective.

In conclusion I wish to say that, while I have discussed the subject from a geometrical viewpoint, there has been an extensive literature to which I am indebted and which it would be impossible to quote in detail. Many of the ideas in geometrical function theory in one variable originated from L. Ahlfors. In the case of high dimensions I should mention in particular the works of H. Schwartz and W. Stoll [8, 10], although they do not seem to have a close contact with the viewpoints envisaged here.

REFERENCES

[1] Borel, A. et J.-P. Serre, Le théorème de Riemann-Roch. *Bull. Soc math. France*, 86,97-136 (1958).

[2] Chern, S. S., Complex analytic mappings of Riemann surfaces, I. *Amer. J. Math.*, 82, 323-337 (1960).

[3] —— The integrated form of the first main theorem for complex analytic mappings in several complex variables. *Annals of Math.*, 71, 536-551 (1960).

[4] —— Some remarks on the quantitative theory of holomorphic mappings, to be submitted to *Comm. Math. Helv.*

[5] Porteous, I. R., *On the Chern classes of algebraic varieties related by a regular algebraic correspondence*, to appear.

[6] Remmert, R., Holomorphe und meromorphe Abbildungen komplexer Räume. *Math. Annalen*, 133, 328-370 (1957).

[7] de Rham, G., *Variétés différentiables*. Paris, 1955.

[8] SCHWARTZ, Marie-Hélène. Formules apparentées à la formule de Gauss-Bonnet pour certaines applications d'une variété à n dimensions dans une autre. *Acta Math.*, 91, 189-244 (1954).

[9] STERNBERG, S. and R. G. SWAN, On maps with non-negative Jacobian. *Mich. Math. J.*, 6, 339-342 (1959).

[10] STOLL, W., Die beiden Hauptsätze der Wertverteilungstheorie bei Funktionen mehrerer komplexer Veränderlichen. Teil I: *Acta Math.*, 90, 1-115 (1953); Teil II: *Acta Math.*, 92, 55-169 (1954).

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