QUADRIGS IN A UNITARY SPACE

QUADRICS IN A UNITARY SPACE

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The study of quadratic forms treats quadrics with centers at the origin. Other quadrics with or without a center are also of some interest. In this paper we shall study the analogues of what was done in [1] for quadrics in an n-dimensional complex unitary space. There are facts in the complex case which are not found in the real case.

1. Definitions and notations: Let A be an $(n+1)$ — *by* — $(n+1)$ Hermitian matrix and $\xi_1 \in E_n$, a unitary space of dimension n. Here $\xi_1 = (x_1, \ldots, x_n)$ and to ξ_1 corresponds the homogeneous form $\xi = (x_1, \ldots, x_n, 1)$. We shall use ξ for both forms whenever there is no confusion. It is convenient to define Q to be the n — *by* — n matrix obtained by eliminating the last row and the last column of A. We consider complex quadrics that can be written in the form $(A\xi, \xi) = 0$, where $\xi = (x_1, \ldots, x_n, 1)$. For example

$$ax\bar{x} + by\bar{y} + h\bar{x}y + \bar{h}x\bar{y} + p\bar{x} + \bar{p}x + q\bar{y} + \bar{q}y + d = 0$$

can be written as

$$(x \quad y \quad 1) \begin{pmatrix} a & \bar{h} & \bar{p} \\ h & b & \bar{q} \\ p & q & d \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \\ 1 \end{pmatrix} = 0 .$$

The choice of a Hermitian matrix has been made in order that the quadrics in certain cases can be expressed geometrically. As an example we discuss the sphere $|\xi - \alpha| = r$, where ξ, $\alpha \in E_n$, α is fixed, r is a positive number, and ξ is variable. This sphere is written as

$$(1.1) \qquad (\xi - \alpha, \xi - \alpha) = r^2 .$$

Let $\xi = (x_1, \ldots, x_n)$ and $\alpha = (a_1, \ldots, a_n)$. Then (1. 1) can be written as

$$(x_1 \ldots x_n \quad 1) \begin{pmatrix} 1 & & 0 & \bar{a}_1 \\ \cdot & & \cdot & \cdot \\ & \cdot & & \cdot \\ 0 & & \cdot & \cdot \\ & & 1 & \bar{a}_n \\ a_1 & \ldots & a_n & \sum a_i \bar{a}_i - r_2 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \cdot \\ \cdot \\ \cdot \\ \bar{x}_n \\ 1 \end{pmatrix} = 0 .$$

A direction δ in a unitary space is defined to be the difference of two vectors. Thus when two vectors ξ and η are expressed in homogeneous form, the homogeneous form of δ is $\delta = (l_1, \ldots, l_n, 0)$.

We shall define a complex straight line to be of the form

$$\xi = \eta + t\delta ,$$

where $\xi, \eta \in E_n$, δ is a complex direction, and t is a complex variable. For convenience we shall denote the variable vector ξ by $(x_1, \ldots, x_n, 1)$, the fixed vector η by $(h_1, \ldots, h_n, 1)$, and δ by $(l_1, \ldots, l_n, 0)$, but whenever it is convenient we use the notations $\xi = (x_1, \ldots, x_n)$ and $\delta = (l_1, \ldots, l_n)$.

2. Intersection of a line and a quadric: The set of equations

$$\begin{cases} (A\xi, \xi) = 0 \\ \xi = \eta + t\delta \end{cases}$$

gives the intersection of the quadric and the line. This set implies

$$(2.\ 1) \qquad (A\eta, \eta) + \bar{t}(A\eta, \delta) + t(A\delta, \eta) + t\bar{t}(A\delta, \delta) = 0 .$$

It is easily seen that $(A\delta, \delta) = (Q\delta, \delta)$. We shall denote the real part of a complex number z by Rz. Then (2. 1) can be written as

$$(2.\ 2) \qquad (Q\delta, \delta)\, t\bar{t} + 2R[\overline{(A\eta, \delta)}\, t] + (A\eta, \eta) = 0 .$$

The following cases may occur:

(a) Let $(Q\delta, \delta) \neq 0$. Then the image of all solutions of (2. 2) in the complex plane is a circle. That is, t satisfies the equation

$$\left| t + \frac{(A\eta, \delta)}{(Q\delta, \delta)} \right|^2 = \left| \frac{(A\eta, \delta)}{(Q\delta, \delta)} \right|^2 - \frac{(A\eta, \eta)}{(Q\delta, \delta)} .$$

In this case we call the intersection *circular*. The vectof corresponding to the center of this circle plays the part or the midpoint in the real case. We shall call this point the mean of the intersections. Note that the image of t may be imaginary, but its center has always a real image.

(b) Let $(Q\delta, \delta) = 0$, but $(A\eta, \delta) \neq 0$. Then (2. 2) is of the form

$$2R\,[\overline{(A\eta, \delta)}\,t] + (A\eta, \eta) = 0 \ , \tag{2. 3}$$

and the image of the solutions of (2. 3) in the complex plane is a line. In this case we call the intersection *linear*.

(c) Let $(Q\delta, \delta) = 0$, $(A\eta, \delta) = 0$, and $(A\eta, \eta) \neq 0$. Then the line does not intersect the quadric.

(d) Let (2. 2) be an identity. Then the line lies on the quadric.

3. Properties of solutions of (2. 2): When $(Q\delta, \delta) \neq 0$, then (2. 2) can be written as

$$t\bar{t} + \frac{(A\eta, \delta)}{(Q\delta, \delta)}\,\bar{t} + \frac{\overline{(A\eta, \delta)}}{(Q\delta, \delta)}\,t + \frac{(A\eta, \eta)}{(Q\delta, \delta)} = 0 \ . \tag{3. 1}$$

The following facts are true for the solutions of (3. 1):

I.

$$\max |\,t\,| \min |\,t = \left| \frac{-(A\eta, \eta)}{(Q\delta, \delta)} \right| \ .$$

II. There is a solution t' of (3. 1) for which

$$|\,t'\,|^2 = \max |\,t\,| \min |\,t\,| \ .$$

III. To any solution t_1 of (3. 1) there corresponds a unique solution t_2 of (3. 1) such that

$$|\,t_1\,| \cdot |\,t_2\,| = |\,t'\,|^2 \ .$$

IV. For any t_1 and t_2 satisfying III, there are two other solutions t_3 and t_4 of (3. 1) which also satisfy III, and

$$(1/2)\,(t_1 + t_2 + t_3 + t_4) = -\frac{(A\eta, \delta)}{(Q\delta, \delta)} \ .$$

V. The four solutions mentioned in IV satisfy

$$\sum_{i=1}^{4} | t_i |^2 = 4 \left[\frac{| (A\eta, \delta) |^2}{(Q\delta, \delta)^2} - \frac{(A\eta, \eta)}{(Q\delta, \delta)} \right],$$

a special case of theorem 1 of [2].

VI. There are two real solutions of (3. 1), say h_1 and h_2, and two pure imaginary solutions, say ik_1 and ik_2 such that

$$h_1 h_2 = k_1 k_2 = - \frac{(A\eta, \eta)}{(Q\delta, \delta)},$$

and

$$h_1^2 + h_2^2 + k_1^2 + k_2^2 = 4 \left[\frac{| (A\eta, \delta) |^2}{(Q\delta, \delta)^2} - \frac{(A\eta, \eta)}{(Q\delta, \delta)} \right].$$

In the case of linear intersection we also have some interesting facts about the solutions of

$$2R[\overline{(A\eta, \delta)}\, t] + (A\eta, \eta) = 0 . \tag{3. 2}$$

VII. This equation has a unique real solution h, and a unique pure imaginary solution ik, which satisfy

$$\frac{1}{h^2} + \frac{1}{k^2} = \frac{1}{\min | t |^2}.$$

VIII. There is a solution t of (3. 2) such that

$$\frac{1}{| h |} + \frac{1}{| k |} = \frac{\sqrt{2}}{| t |}.$$

IX. There is a solution t of (3. 2) such that

$$| h |^2 + | k |^2 = 4 | t |^2 .$$

X. For any solution t_1 of (3. 2) there is a solution t_2 of (3. 2) such that

$$\frac{1}{| t_1 |^2} + \frac{1}{| t_2 |^2} = \frac{1}{\min | t |^2}.$$

4. Theorem: The locus of the means of intersections of a quadric with all lines of fixed direction δ is a plane, i.e., satisfies a first degree equation in $x_1, \dots, x_n$. We call this plane a diametral plane corresponding to δ.

Proof: Let $(Q\delta, \delta) \neq 0$ so that there will be a circular intersection [see 2 *(a)*], and let η be the mean of intersections. Then $\xi = \eta + t\delta$ is a line which cuts $(A\xi, \xi) = 0$ in a circular intersection with mean η. In this case we have $(A\eta, \delta) = 0$. Thus the locus of η is the plane $(A\xi, \delta) = 0$.

5. Corollary: If the diametral plane is orthogonal to the corresponding direction δ, then δ is an eigenvector of Q. In this case the diametral plane is called the plane of symmetry corresponding to δ.

Proof: We easily see that $(A\xi, \delta) = 0$ is equivalent to $(A\delta, \xi) = 0$, and can be written as $(Q\delta, \xi) + M = 0$, where M is a constant. Since the direction normal to $(A\delta, \xi) = 0$ is $Q\delta$, and δ lies in this direction we have

$$Q\delta = k\delta \ ,$$

which proves the corollary.

6. Centers of a quadric: A vector $\eta = (x_{10}, \ldots, x_{no})$ is a center of a quadric $(A\xi, \xi) = 0$ if η is the mean of circular intersections of all lines through it.

For the center let

$$\xi = \eta + t\delta \tag{6. 1}$$

be a line through η, where $\delta = (l_1, \ldots, l_n) = (x_1 - x_{10}, \ldots, x_n - x_{no})$. The intersection of (6. 1) and $(A\xi, \xi) = 0$ is obtained from the equation

$$(Q\delta, \delta)\, t\bar{t} + 2R[\bar{t}\,(A\eta, \delta)] + (A\eta, \eta) = 0 \ , \qquad (Q\delta, \delta) \neq 0$$

[see 2 *(a)*]. For η to be the mean of intersections we must have

$$(A\eta\,, \delta) = 0 \ . \tag{6. 2}$$

Suppose the matrix A has the form

$$A = \begin{pmatrix} h_{11} \cdots \bar{h}_{1n} & \bar{p}_1 \\ \vdots \quad\ \vdots & \vdots \\ h_{1n} \cdots h_{nn} & \bar{p}_n \\ p_1 \ \cdots\ p_n & d \end{pmatrix} .$$

Then (6. 2) can be written as

$$(x_{10} \cdots x_{n0}\ 1) \begin{pmatrix} h_{11} \cdots \bar{h}_{1n} & \bar{h}_1 \\ \cdot \quad\quad \cdot & \cdot \\ \cdot \quad\quad \cdot & \cdot \\ \cdot \quad\quad \cdot & \cdot \\ h_{1n} \cdots h_{nn} & \bar{p}_n \\ p_1 \cdots p_{nn} & d \end{pmatrix} \begin{pmatrix} \bar{l}_1 \\ \cdot \\ \cdot \\ \cdot \\ \bar{l}_1 \\ 0 \end{pmatrix} = 0 \, .$$

This implies that

$$\alpha = (p_1 + \textstyle\sum_{i=1}^n h_{1e}\, x_{i0}\,, \ldots, p_n + \sum_{i=1}^n \bar{h}_{in}\, x_{i0})$$

is orthogonal to δ for all ξ satisfying $(A\xi, \xi) = 0$. Either α is fixed and $(A\xi, \xi) = 0$ is a double plane orthogonal to α at η, or $\alpha = 0$, that is,

$$p_j + \textstyle\sum_{i=1}^n h_{ji}\, x_{i0} = 0\,,\ j = 1, \ldots, n\ .$$

In other words $(x_{10}, \ldots, x_{n0})$ is the solution of

$$(x_1 \cdots x_n) \begin{pmatrix} h_{11} \cdots \bar{h}_{1n} \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ h_{1n} \cdots h_{nn} \\ p_1 \cdots p_n \end{pmatrix} = (0 \ldots 0)\ .$$

The problem of existence of a center is the problem of n linear equations. We shall leave the discussion of this problem to the reader.

7. Vertex of a quadric: We shall only discuss the case when the quadric has no center. A necessary condition for lack of center is that the rank of Q be less than n. Of course this condition is not sufficient.

Let $\alpha_1, \ldots, \alpha_k$ be orthonormal eigenvectors of Q corresponding to nonzero eigenvalues of Q. Let

$$(7.\ 1) \qquad (A\alpha_i, \xi) = 0\ , \quad i = 1, \ldots, k$$

be planes of symmetry corresponding to α_i. It is well known that the quadratic terms of the quadric constitute a linear combination of terms, each of which is the product of the first degree part of $(A\alpha_i, \xi)$ and its cojugate. Thus the $k + 1$ equations

$$(A\xi, \xi) = 0\ , \quad (A\alpha_i, \xi) = 0\ , \quad i = 1, \ldots, k$$

give vertices. We leave the discussion of different cases to the reader.

8. Tangent plane: The idea of tangency is slightly different from the real case. A line $\xi = \eta + t\delta$ is said to be tangent to a quadric $(A\xi, \xi) = 0$ if its circular intersection with the quadric is a single point, namely its mean. If a line changes in such a way that it remains tangent to a quadric at a fixed point, then its locus is a plane called the tangent plane.

To prove this fact, i.e., to obtain this tangent plane we shall study (2. 2). For the circular intersection to be only its mean we must have

$$\frac{|(A\eta, \delta)|^2 - (A\eta, \eta)(Q\delta, \delta)}{(Q\delta, \delta)} = 0 \quad .$$

[see 2 *(a)*]. Here η is on the quadric, so that $(A\eta, \eta) = 0$. Therefore for tangency we have

$$(A\eta, \delta) = 0 \; . \tag{8. 1}$$

Let $\eta = (x_{10}, \ldots, x_{n0})$, $\delta = (l_1, \ldots, l_n)$, and the matrix A be the same as in 6. Then (8. 1) implies that

$$(h_{11}\, x_{10} + \ldots + h_{nn}\, x_{n0} + p_n)\, \bar{l}_1 + \ldots + (\bar{h}_{in}\, x_{10} + \\ + \ldots + h_{nn}\, x_{n0} + p_n)\, \bar{l}_n = 0.$$

This implies that the vector

$$\alpha = (h_{11}\, x_{10} + \ldots + h_{1n}\, x_{n0} + p_1, \ldots, \bar{h}_{1n}\, x_{10} + \bar{h}_{2n}\, x_{20} + \\ + \ldots + h_{nn}\, x_{n0} + p_n)$$

is perpendicular to δ and therefore to the locus of δ. Thus δ is in a plane and the tangent plane is $(A\eta, \xi) = 0$.

9. Pole and polar: Let $\eta = (x_{10}, \ldots, x_{n0})$ be fixed and δ any direction. Then

$$\xi = \eta + t\delta \tag{9. 1}$$

is any line through η. The intersection of this line and $(A\xi, \xi) = 0$ is obtained by (2. 2). Let $(Q\delta, \delta) \neq 0$ so that we have circular intersections.

For any ξ_1 and ξ_2 satisfying

$$\xi = \eta + t\delta \,, \qquad (A\xi, \xi) = 0$$

and corresponding t_1 and t_2 of (2. 2) for which $|t_1| \cdot |t_2| = |t'|^2$ [see 3 III], there is a point ξ called the harmonic cojugate of η with respect to ξ_1 and ξ_2, so that the parameter t corresponding to this ξ satisfies the harmonic relation

$$\frac{2}{|t|} = \frac{1}{|t_1|} + \frac{1}{|t_2|} \,.$$

This implies that the image of t in the complex plane is the polar of the origin with respect to the image of t_0, defining the intersection of the line $\xi = \eta + t_0 \delta$ and $(A\xi, \xi) = 0$. Thus

$$t = -\frac{(A\eta, \eta)}{(A\eta, \delta)} \,, \qquad (A\eta, \delta) \neq 0 \,.$$

If $(A\eta, \delta) = 0$, then the polar is at infinity. Substituting this t in (9. 1) we get

$$\overline{(A\eta, \delta)}\, \xi = \overline{(A\eta, \delta)}\, \eta - (A\eta, \eta)\, \delta \,.$$

From the inner product of both sides with $A\eta$ we get

$$\overline{(A\eta, \delta)}\, (A\eta, \xi) = 0 \,,$$

and since $(A\eta, \delta) \neq 0$ we get $(A\eta, \xi) = 0$, the equation of a plane called the polar of η with respect to the quadric.

Note that if η is on the quadric, then this plane becomes the tangent plane.

In this paper we have discussed only the properties of a quadric in any location. The problem of transformation to the most convenient position is well known, and we made use of it in section 7.

REFERENCES

1. A. R. Amir-Moéz, A. L. Fass, Quadrics in R_n. *Amer. Math. Monthly*, vol. 67, nº 7, pp. 632-636 (1960).
2. A. R. Amir-Moéz, Some Equalities in a Unitary Space Leading to Equalities Concerning Singular Values of Sets of Matrices. *Mathematische Annalen*, 135 Band, 5 (Schluss) Heft, 1958, S. 388-390.

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