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Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **8 (1962)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **24.05.2024**

A SURVEY OF COBORDISM THEORY[1]

by J. MILNOR

This paper will start out with a discussion of known results and then will taper off into a discussion of unsolved problems.

The theory of cobordism was initiated by L. Pontrjagin and V. A. Rohlin [10, 12]. It came of age with the work of R. Thom [17]. The basic question in this theory is the following. Let $\mathscr{M}$ be some class of compact manifolds. Given $V \in \mathscr{M}$ how can one decide whether or not V is the boundary of some other manifold in $\mathscr{M}$? Of course a necessary condition is that V itself must be a *closed* manifold: that is the boundary ∂V must be vacuous.

1. The classical cobordism groups N_k and Ω_k.

As a first illustration of this problem let $\mathscr{D}$ denote the class of all compact differentiable manifolds. The manifolds $V \in \mathscr{D}$ need not be connected or orientable, and are allowed to have boundaries.

THEOREM 1 (Pontrjagin, Thom). — *A closed* k*-dimensional manifold* V $\in \mathscr{D}$ *is the boundary of some* (k + 1) *-dimensional manifold in* $\mathscr{D}$ *if and only if the Stiefel-Whitney numbers* $w_{i_1} \dots w_{i_n}$ [V] *are all zero.*

(Explanation: The Stiefel-Whitney cohomology classes[2]) $w_i \in H^i(V; J_2)$ are defined for example in Steenrod [15]. If $i_{i_1} + \dots + i_n = k$ is any partition of k then the cup product $w_{i_1} \dots w_{i_n}$ is a top dimensional cohomology class. Applying the canonical " integration " homomorphism

$$[V]: H^k(V; J_2) \rightarrow J_2$$

we obtain a " Stiefel-Whitney number " $w_{i_1} \dots w_{i_n}[V] \in J_2$.)

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1) Talk delivered at the Zurich Colloquium on Differential Geometry and Topology, June 1960.

2) The notation J will be used for the integers and J_2 for the integers modulo 2.

The *non-oriented cobordism group* $N_k = H_k(\mathscr{D})$ is constructed as follows. Given two k-manifolds $V, V' \in \mathscr{D}$ the *sum* $V + V'$ will mean the (disjoint) topological sum, provided with a differentiable structure in the obvious way.

Definition. Two closed manifolds $V, V' \in \mathscr{D}$ are *congruent modulo* $\partial\mathscr{D}$ if $V + V'$ is the boundary of some manifold in $\mathscr{D}$. The set of all congruence classes of closed k-manifolds, under the composition operation $+$, forms the required group N_k. We will also use the notation $H_k(\mathscr{D})$ for this group since it is something like a homology group. (The Russian term for " cobordism " is " intrinsic homology ".)

It follows from Theorem 1 that each N_k is a finite abelian group of the form $J_2 \oplus \ldots \oplus J_2$.

The cartesian product operation between differentiable manifolds gives rise to a bilinear pairing

$$N_k \oplus N_l \to N_{k+l} .$$

Thus the graded group $N_* = (N_0, N_1, \ldots)$ has the structure of a graded ring.

THEOREM 2 (Thom). — *The non-oriented cobordism ring* N_* *has the structure of a polynomial algebra*

$$J_2 [X_2, X_4, X_5, X_6, X_8, X_9, \ldots]$$

with one generator $X_k \in N_k$ *for each dimension which is not of the form* $2^m - 1$.

If k is even then the real projective k-space can be taken as generator. For k odd generators have been constructed by Dold [4].

Thom's proof of Theorems 1 and 2 involves a brilliant mixture of algebra and geometry. A key step in the argument is his proof that N_k is isomorphic to a certain homotopy group. I will not try to give details.

Next consider the class $\mathscr{D}_0$ consisting of all *oriented* compact differentiable manifolds.

THEOREM 1'. — *A closed manifold in* $\mathscr{D}_0$ *is the boundary of a manifold in* $\mathscr{D}_0$ *if and only if both its Stiefel-Whitney numbers and its Pontrjagin numbers are zero.*

This result is due to Pontrjagin, Thom, Milnor, Averbuh, and Wall. (See [2, 9, 19].) For the definition of the Pontrjagin numbers $p_{i_1} \dots p_{i_n} [V] \in J$ the reader is refered to Hirzebruch [6]. These numbers are defined only if the dimension k is a multiple of 4.

The *oriented cobordism ring* $\Omega_* = H_*(\mathscr{D}_o)$ is defined as follows. For $V \in \mathscr{D}_o$ let $-V$ denote the same manifold V with the opposite orientation. We will say that

$$V \equiv V' \quad (mod\ \partial \mathscr{D}_o)$$

if $(-V) + V'$ is the boundary of some manifold in $\mathscr{D}_o$. As an example, for any closed manifold V we have $V \equiv V \pmod{\partial \mathscr{D}_o}$ since

$$(-V) + V \approx \partial (V \times I)$$

where I denotes the unit interval. The set of all such congruence classes form the required group Ω_k. Again the cartesian product operation makes $\Omega_* = (\Omega_0, \Omega_1, \dots)$ into a graded ring.

It follows from Theorem 1′ that Ω_k is a finitely generated group of the form

$$J \oplus \dots \oplus J \oplus J_2 \oplus \dots \oplus J_2$$

where infinite cyclic summands can occur only if $k \equiv 0 \pmod 4$.

THEOREM 2′. — *The ring Ω_*, modulo the ideal consisting of 2-torsion elements, is a polynomial ring* $J[Y_4, Y_8, Y_{12}, \dots]$ *with one generator in each dimension divisible by* 4.

The complex projective space of real dimension $4m$ can be taken as generator for $m = 1, 2, 3$. However a different generator is needed in dimension 16.

For a description of the 2-torsion in Ω_* the reader is referred to Wall's paper.

2. MANIFOLDS WITH X-STRUCTURE.

In this section we will define the concept of an " X-structure " on the tangent bundle of a differentiable manifold; and study the corresponding cobordism theory.