4. Preliminary Lemmas

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R^n, and $(n-1)$-spheres oriented with orientations induced by their interiors.

Symbols c^{n-1}, g^{n-1}, ... denote oriented $(n-1)$-cycles in R^n; D^{n-1}, V^{n-1}, ... denote $(n-1)$-spheres in R^n. E^n denotes a closed solid n-sphere in R^n, and the boundary of E^n is denoted by S^{n-1}. η^n denotes a closed n-cell in R^n and the boundary of η^n is denoted by σ^{n-1}.

In this paper η^n is assumed to be the image of E^n under homeomorphism θ, and η^n and σ^{n-1} obtain their orientations from E^n and S^{n-1} respectively.

3. The Turning Index

Let c^{n-1} be an $(n-1)$-cycle in R^n and g a continuous map of c^{n-1} into R^n having no fixed point. Let D^{n-1} be an $(n-1)$-sphere with center 0, called a *direction sphere* [2]. Let c^{n-1} be mapped on D^{n-1} as follows. To a point $c \varepsilon c^{n-1}$ there corresponds a point $d \varepsilon D^{n-1}$ such that the line segment from 0 to d has the same sense and direction as that from c to $g(c)$. The resulting $(n-1)$-cycle g^{n-1} on D^{n-1} is called, in the sequel, *the $(n-1)$-cycle g^{n-1} resulting from g applied to c^{n-1}*, and the degree of the resulting map, that is, the multiple of D^{n-1} which is homologous to g^{n-1} (which is clearly independent of the radius of D^{n-1} and the location of 0) is called the *turning index* of c^{n-1} under g.

If p is a point not on c^{n-1}, the *index of p relative to c^{n-1}* is defined as the turning index of the map which maps every point of c^{n-1} into p. (For odd n, this is the negative of the corresponding definition given in [3], as shown by Theorem 1.5, page 105).

4. Preliminary Lemmas

Lemma 1. *Let g and h be two continuous maps into R^n of an $(n-1)$-cycle c^{n-1}, such that neither leaves any point of c^{n-1} fixed, and, for no point $c \varepsilon c^{n-1}$ are the directions from c to $g(c)$ and from c to $h(c)$ exactly opposite. Then the turning indices of c^{n-1} under g and h are equal.*

Proof. For each $c \varepsilon c^{n-1}$, the directions of the two vectors $\overline{c, g(c)}$ and $\overline{c, h(c)}$ are not opposite and hence, if not identical, determine a 2-plane P in which they make an angle of less than π radians. As a parameter p varies from 0 to 1, let the direction of $\overline{c,h}(c)$ change in P so that the angle between the two vectors $\overline{c,h}(c)$ and $\overline{c,g}(c)$ decreases uniformly to zero while their lengths remain fixed. If the angle is zero at the start, no change in direction takes place. For each value of p, $0 \leqq p \leqq 1$, the corresponding mapping as determined above in the definition of turning index, maps c^{n-1} on the direction sphere D^{n-1}, and the result, as p varies from 0 to 1, is to deform the $(n-1)$-cycle h^{n-1} on D^{n-1} resulting from h applied to c^{n-1} into the $(n-1)$-cycle g^{n-1} resulting from g applied to c^{n-1}. Hence h^{n-1} is homologous to g^{n-1}, and therefore to the same multiple of D^{n-1}, so that the turning indices under consideration are equal. Thus Lemma 1 is proved.

LEMMA 2. *Let g be a continuous map into R^n of an $(n-1)$-cycle c^{n-1},* such that c^{n-1} and $g(c^{n-1})$ *are contained in different half-spaces into which R^n is separated by some $(n-1)$-plane. Then the turning index of c^{n-1} under g is zero.*

Proof. Since the $(n-1)$-cycle g^{n-1} resulting from g applied to c^{n-1} is clearly entirely on one hemisphere of D^{n-1}, we conclude that c^{n-1} cannot be homologous to any multiple of D^{n-1} other than zero. Thus Lemma 2 is proved.

LEMMA 3. *Let σ^{n-1} be the boundary of a closed n-cell $\eta^n \subset R^n$. Let e be a point in the inside of σ^{n-1}. Then the index of e relative to σ^{n-1} is 1 or -1.*

While this result is given in [3], page 109, Theorem 4.1, the following proof is given as shorter and obtained independently.

Proof. Let η^n and σ^{n-1} be respectively the homeomorphic images (under homeomorphism θ) of the closed solid n-sphere E^n with boundary S^{n-1}, i.e., $\eta^n = \theta(E^n)$ and $\sigma^{n-1} = \theta(S^{n-1})$. By use of the invariance of regionality, it is easy to show that $\eta^n = \theta(E^n)$ contains no point outside σ^{n-1} and contains every point inside σ^{n-1}.

Let V^{n-1} be an $(n-1)$-sphere with center at e, so small that V^{n-1} and its interior are inside σ^{n-1}, hence composed of points of η^n. Let $\beta^{n-1} = \theta^{-1}(V^{n-1})$ and $d = \theta^{-1}(e)$.

For each point $b \varepsilon \beta^{n-1}$ let the half-line beginning at d and passing through b intersect S^{n-1} at b'.

Now, for every t, with $0 \leqq t \leqq 1$, let $\beta^{n-1}(t)$ be the $(n-1)$-cycle determined as follows. For each point $b \varepsilon \beta^{n-1}$ there corresponds a point $b(t)$ of $\beta^{n-1}(t)$ on the closed segment from b to b' such that the distance from b to $b(t)$ is t times the distance from b to b'.

Let $V^{n-1}(t) = \theta[\beta^{n-1}(t)], \qquad 0 \leqq t \leqq 1.$

As t varies from 0 to 1, the cycle $V^{n-1}(t)$ undergoes a deformation from initial position $V^{n-1}(0) = V^{n-1}$ to final position $V^{n-1}(1)$. Since $V^{n-1}(1)$ is on σ^{n-1}, there is an integer x such that

$$V^{n-1}(1) \sim x\,\sigma^{n-1} \qquad \text{on } \sigma^{n-1}, \tag{1}$$

where $\sim$ stands for " is homologous to ".

For each t, let $k(t)$ be the mapping which maps every point of $V^{n-1}(t)$ into e, and let V^{n-1} serve as the direction sphere. As t varies from 0 to 1, the $(n-1)$-cycle $k^{n-1}(0)$ resulting from $k(0)$ applied to V^{n-1} is deformed on the direction sphere V^{n-1} into the $(n-1)$-cycle $k^{n-1}(1)$ resulting from $k(1)$ applied to $V^{n-1}(1)$. Thus these two $(n-1)$ cycles are homologous on V^{n-1}. Therefore the index of e relative to V^{n-1} equals the index of e relative to $V^{n-1}(1)$. However, since $k(0)$ maps every point of V^{n-1} into e, we derive that ([4], page 92)

(2) the index of e relative to $V^{n-1}(1) = (-1)^n$.

Let y be the index of e relative to σ^{n-1}. From (1) we infer that xy is the index of e relative to $V^{n-1}(1)$. Hence, by (2), $xy = (-1)^n$. Consequently, $y = 1$ or $y = -1$. Thus Lemma 3 is proved.

LEMMA 4. *If a continuous map f of a closed n-cell $\eta^n \subset R^n$ into R^n has no fixed point, then the turning index of the boundary σ^{n-1}* of *η^n under f is zero.*

Proof. Let, as in the proof of Lemma 3, $\eta^n = \theta(E^n)$ and $\sigma^{n-1} = \theta(S^{n-1})$ be respectively the images under the homeomorphism θ of the closed solid n-sphere E^n and its boundary S^{n-1}.

Let u be the center of S^{n-1}. Since f has no fixed point, it is clear that we can choose $d > 0$ so small that a closed solid n-sphere H_d^n of radius d with center at $\theta(u)$ is entirely in η^n, and H_d^n and its image $f(H_d^n)$ are contained in different half-spaces into which R^n is separated by some $(n-1)$-plane.

Now, let S^{n-1} undergo a deformation by uniform radial shrinking toward u till it reaches a position S_2^{n-1} whose image σ_2^{n-1} under θ is contained in the interior of H_d^n. By means of θ, there results a deformation of σ^{n-1} into σ_2^{n-1} which by means of the mapping f induces a deformation, on the direction sphere, of the $(n-1)$-cycle f^{n-1} resulting from f applied to σ^{n-1} into the $(n-1)$-cycle f_2^{n-1} resulting from f applied to σ_2^{n-1}.

Thus the turning index of σ^{n-1} under f equals the turning index of σ_2^{n-1} under f, which by Lemma 2 equals zero. Thus Lemma 4 is proved.

5. The Theorems

Theorem 1. *Let $\eta^n \subset R^n$ be a closed n-cell and f a continuous mapping of η^n into R^n such that f maps the boundary σ^{n-1} of η^n into η^n. Then f has at least one fixed point.*

Proof. Assume no fixed points. Let, as in the case of Lemma 3, η^n and σ^{n-1} be respectively the images (under the homeomorphism θ) of the closed solid n-sphere E^n with boundary S^{n-1}, i.e., $\eta^n = \theta(E^n)$ and $\sigma^{n-1} = \theta(S^{n-1})$.

Let u be the center of S^{n-1}. Consider the mapping f' of σ^{n-1} which maps every point $\sigma \in \sigma^{n-1}$ into the point $\theta(u)$. Since f' is the mapping which appears in the definition of the index of $\theta(u)$ relative to σ^{n-1}, we see by Lemma 3 that the turning index of σ^{n-1} under f' is non-zero.

By hypothesis, $f(\sigma) \in \eta^n$ for every $\sigma \in \sigma^{n-1}$. Hence we may deform $f(\sigma^{n-1})$ as follows. As a parameter p varies from 0 to 1, the point σ' moves in η^n along the path $\theta[\overline{\theta^{-1} f(\sigma), u}]$ starting from σ and ending at $\theta(u)$.

For $p = 1$, the above deformation yields the mapping f'. Therefore, the $(n$-1)-cycle resulting from f applied to σ^{n-1} is homologous on the direction sphere (as a consequence of a deformation) to the $(n-1)$-cycle resulting from f' applied to