4. TWO DIFFERENCIAL OPERATORS OF THE ORDERS p AND q.

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the factors $B(\chi)$ and $\Gamma(\chi)$ also in an alternative form. This is done as follows:

Let K designate the familiar matrix

$$K(z) = \begin{bmatrix} 0 & 0 & - & - & - & - & -b_p \\ 1 & 0 & - & - & - & - & -b_{p-1} \\ 0 & 1 & 0 & - & - & - & -b_{p-2} \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ 0 & - & - & - & - & 1 & -b_1 \end{bmatrix} \qquad (3.\ 3)$$

The eigen-values of this are the roots x_i, $i = 1, 2, \ldots, p$ of the equation $B(x) = 0$. If we designate by ξ_i an eigen-vector corresponding to x_i we have

$$K^h \xi_i = x_i^h \xi_i\,, \quad h = 1, 2, \ldots, q\,,$$

and accordingly

$$\Gamma(K)\,\xi_i = \Gamma(x_i)\,\xi_i\,.$$

Thus $\Gamma(x_i)$ is an eigenvalue of the matrix $\Gamma(K)$, and, since the product of the eigenvalues is the determinant of the matrix, we have

$$|\,\Gamma(K)\,| = \prod_{i=1}^{p} \Gamma(x_i)\,.$$

Observing that no factor on the right is zero, and giving to $\Gamma(K)$ its explicit form, we conclude with the result

$$\Big|\sum_{j=0}^{q} c_j(z)\, K^{q-j}(z)\Big| \neq 0 \qquad (3.\ 4)$$

4. Two differencial operators of the orders p and q.

Let the functions $\beta_j(z, \lambda)$ and $\gamma_i(z, \lambda)$ be taken to be polynomials of the degree $(r - 1)$ in $1/\lambda$, thus

$$\beta_j(z, \lambda) = \sum_{\nu=0}^{r-1} \frac{\beta_{j,\nu}(z)}{\lambda^\nu}\,, \quad \beta_{j,0}(z) \equiv b_j(z)\,;\ j = 1, 2, \ldots, p\,,$$

$$\gamma_i(z, \lambda) = \sum_{\nu=0}^{r-1} \frac{\gamma_{i,\nu}(z)}{\lambda^\nu}\,, \quad \gamma_{i,0}(z) \equiv c_i(z)\,;\ i = 1, 2, \ldots, q\,. \qquad (4.\ 1)$$

As has been indicated, the terms of the zeroth degree are to be the coefficients which appear in the formulas (2. 7). The remaining terms, $\beta_{j,\nu}(z)$ and $\gamma_{i,\nu}(z)$, with $\nu \geqq 1$, shall be analytic over the z-region, but beyond that shall be left, for the moment, unspecified. By l and m we shall designate the differential operators

$$\begin{aligned} l &= \sum_{j=0}^{p} \lambda^j \beta_j(z, \lambda) D^{p-j}, \quad \beta_0 \equiv 1, \\ m &= \sum_{i=0}^{q} \lambda \gamma_i(z, \lambda) D^{q-i}, \quad \gamma_0 \equiv 1. \end{aligned} \tag{4. 2}$$

The immediate objective will be to show that the unspecified terms in the formulas (4. 1) can be so chosen as to give the differential form $l(m(u))$ coefficients which differ from those of the form (2. 2) only by terms that are of at least the r^{th} degree in $1/\lambda$.

The k-fold differentiation of $m(y)$ yields the formula

$$D^k m(y) = \sum_{i=0}^{q} \sum_{s=0}^{k} \lambda^i \binom{k}{s} D^s \gamma_i D^{q-i+k-s} y,$$

in which the symbol $\binom{k}{s}$ denotes, as customarily, the coefficient of x^s in the binomial expansion of $(1+x)^k$. On using $i+s$ in place of i as the variable of summation, and observing that the terms which appear to have been gratuitously included are ones to which the value zero are to be assigned, we find that the formula may be written

$$D^k m(y) = \sum_{i=0}^{q+k} \sum_{s=0}^{p} \lambda^{i-s} \binom{k}{s} D^s \gamma_{i-s} D^{q+k-i} y, \qquad k = 0, 1, 2, \ldots, p. \tag{4. 3}$$

From this it follows that

$$l(m(y)) = \sum_{j=0}^{p} \sum_{i=0}^{q} \sum_{s=0}^{p-j} \lambda^{j+i-s} \binom{p-j}{s} \beta_j D^s \gamma_{i-s} D^{n-j-i} y.$$

This formula is again improved by using $i+j$ in place of i as the variable of summation. It becomes, then

$$l(m(y)) = \sum_{i=0}^{n} \lambda^i \Psi_i(z, \lambda) D^{n-i} y, \tag{4.4}$$

with

$$\Psi_i(z, \lambda) = \sum_{j=0}^{p} \sum_{s=0}^{p-j} \lambda^{-s} \binom{p-j}{s} \beta_j D^s \gamma_{i-j-s} \,. \tag{4.5}$$

The functions $\Psi_i(z, \lambda)$, inasmuch as they are combinations of those given in (4. 1), are polynomials in $1/\lambda$. We may therefore write them in the form

$$\Psi_i(z, \lambda) = \sum_{\mu=0}^{r-1} \frac{\psi_{i,\mu}(z)}{\lambda^\mu} + \frac{\psi_{i,r}(z, \lambda)}{\lambda^r} \,. \tag{4.6}$$

A comparison of the terms in like powers of $1/\lambda$ in the relations (4. 5) and (4. 6) yields formulas for the functions $\Psi_{i,\mu}(z)$. Those for which $\mu = 0$ are particularly easy to obtain. On setting $s = 0$ in (4. 5), and replacing β_j and γ_{i-j} by their leading terms b_j and c_{i-j}, we find that

$$\psi_{i,0}(z) = \sum_{j=0}^{p} b_j(z) c_{i-j}(z) \,.$$

Recourse to the relation (2. 8) thus shows that

$$\psi_{i,0}(z) = p_{i,0}(z), \quad i = 1, 2, \ldots, n \,. \tag{4.7}$$

At least to the extent of the leading terms of their coefficients, the forms (2. 2) and (4. 4) are, therefore, the same.

5. A determination of unspecified coefficients.

We propose now to deduce a formula for the general coefficient $\psi_{i,\mu}(z)$ in (4. 6) by selecting the multiplier of the appropriate power of $1/\lambda$ from the formula (4. 5). To begin with, it follows from the relations (4. 1) that

$$\beta_j D^s \gamma_{i-j-s} = \sum_{\mu=0}^{2r-2} \sum_{k=0}^{\mu} \lambda^{-\mu} \beta_{j,k} D^s \gamma_{i-j-s,\mu-k}$$