3. The new curricula

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of suburban school systems adopted School Mathematics Study Group materials, starting with the 7th year of school, and have developed their own materials for the first six years. They are now facing the very serious problem that by the time their students have studied modern mathematics (in an elementary version) for the first six years, they will find the " frightening " new ideas of the 7th and 8th years much too easy, and hence these schools will find the modernized curricula terribly old-fashioned.

3. The new curricula

The most striking feature of the 21 reports is the degree of similarity in the proposals for including new topics of mathematics.

There are four areas of modern mathematics that are recommended by a majority of the reports. These are elementary set theory, an introduction to logic, some topics from modern algebra, and an introduction to probability and statistics. Equally frequent is a mention of the necessity for modernizing the language and conceptual structure of high school mathematics.

Perhaps the most frequently mentioned topic is that of elementary set theory. The concept of a set, as well as the operation of forming unions, intersections, and complements, constitute a common conceptual foundation for all of modern mathematics. It is therefore not surprising that almost all nations favoring any modernization of the high school curriculum have advocated an early introduction to these simple, basic ideas. An attractive feature of this topic is that in a relatively short time a student may be given a feeling of the spirit of modern mathematics without involving him in undue abstraction.

It should, however, be noted that in most cases only an elementary introduction of this topic is recommended. For example, the usual " next " topic in developing set theory is that of cardinality. Only three nations have suggested this as a possible topic for inclusion in the secondary curriculum.

The introduction of elementary symbolic logic may be justified on grounds quite similar to that of the introduction of sets. Indeed, the most elementary structures in the two subjects, Boolean algebra and the propositional calculus, are isomorphic. It is, therefore, not surprising that in several countries these topics are studied more or less simultaneously, exploiting the various possible ways of setting up isomorphisms between the systems.

Of course, logic plays a strange dual role in the mathematics curriculum, in that logical reasoning is an underlying feature of all mathematical arguments, and at the same time modern symbolic logic is an interesting topic in its own right. After many centuries of making free use of logic, without careful examination of its basic principles, the mathematician has turned around and made logic one of the branches of mathematics. It should again be noted that in most cases only very elementary principles of logic have been suggested for study in the high school curriculum.

The status of probability and statistics is entirely different from that of logic and sets. The introduction of these subjects into the high school curriculum is proposed usually on the basis of their inherent attractiveness and importance, rather than their instrumental use in other branches of mathematics. In almost all cases both probability and statistics were advocated, usually closely tied together. I shall follow the convention that under the heading of " probability " a branch of pure mathematics is meant, while " statistics " describes a branch of applied mathematics. If this view is accepted, we must see here both the most widely recommended subject in pure mathematics and the only widely recommended subject in applied mathematics, for inclusion in high school education.

I would like to suggest that the extent to which probability theory is to be taught in high school should be one of the topics of discussion following this report to the Congress. Probability theory recommends itself as a very attractive branch or pure mathematics because it is so easy to give examples, from everyday experience, involving probabilistic computations. Therefore, the student is challenged to combine mathematical rigor and intuition.

However one may consider introducing probability theory from a purely classical point of view, in which one deals with equally likely events and defines probability simply as a ratio of favorable outcomes to total number of outcomes. In this case, probability problems reduce to problems of counting or combinatorics. There is no doubt that such simple combinatorial problems are well within the grasp of the average high school student and, indeed, such topics have long been included in high school algebra courses. In many of the reports sent to me it was not clear whether the probability theory advocated goes beyond such elementary computations.

To capture any of the spirit of modern probability theory, it is necessary to introduce the concept of a measure space and to define probabilities of various events in terms of measures of subsets. While anything like a full treatment of measure theory is much too difficult for high school students, a number of experiments have shown the possibility of doing this for discrete situations, or even more restricted, for finite sets. Since the normal problems familiar to high school students deal only with a finite number of possible outcomes, this formulation of the foundations of probability theory corresponds particularly closely to the students' every-day experience. Recommendations for such a very elementary treatment of probabilistic measure theory are contained in four reports.

While a majority of reports contained a suggestion that some topics from modern algebra should be chosen, there was considerably less agreement as to what this choice should be. Basically, there seems to be a split between the advocates of teaching topics from algebraic systems (groups, rings, and fields) and those who advocate linear algebra. In a few cases, both types of topics were suggested, but usually the lack of time in high school curricula prevents the introduction of a very sizable amount of modern algebra.

It seems to me that the motivation for these two types of topics have many common features. The introduction, on an axiomatic basis, of any modern algebra has the very healthy feature of removing the common misconception that axiomatics is somewhat closely tied with geometry. I recall once having a student who told me that, in his experience, the difference between algebra and geometry was that "in geometry you proved things, while in algebra somebody just told you what to do". Certainly, this objective can be equally well achieved by introducing as one's basic axiomatic system either that of a group or that of a vector space.

In addition to this, either linear algebra or algebraic systems have the advantage of giving deeper insight into certain structures known to the students for other reasons. Linear algebra, of course, has many applications to geometry, while algebraic structures arise as generalizations of one's experience with numbers.

The usual argument given for the introduction of groups, rings, and fields is that this is the only way one can bring about a true understanding of the nature of our number system. Attempts to prove to the student simple rules, such as those governing the operations with fractions, often fail because both the basic assumptions and the results to be proved are too familiar to the student. However, by moving to an abstract axiomatic system, the student is forced to abandon his intuition and rely on mathematical rigor in his proof.

It may certainly be said, if one wishes to introduce one example of an axiomatic system in modern algebra, that the simplest and most universally useful one is that for a group. It also has the attractive feature that, in addition to being applicable to many groups of numbers well known to the student, one can introduce such simple and interesting examples as the symmetries of a simple geometric object (e.g., a square).

A study of vector spaces, of course, is much more difficult than the study of a simple system such as a group. I have not seen any suggestion of studying vector spaces over an arbitrary field. However, there were a number of suggestions for studying a vector space over the real numbers. Here much of the difficulty is removed by relying on the student's intuitive understanding of the underlying field. Presumably, the major motivation for this line of inquiry is that it helps to clarify much of what the student was forced to learn before. For example, it can be used to give new insight into the meaning of the solutions of simultaneous equations. Equally important, of course, are the numerous applications of linear algebra to geometry. While geometry can be used to motivate linear algebra, linear algebra, in turn can be used to make the nature of geometric transformations more clearly understood.

I must now mention a few topics which occurred occasionally amongst the recommendations, though these seem to be topics not nearly so widely accepted. These include some modern topics in geometry, the study of equivalence and order relations, cardinal numbers, and an introduction to elementary topology. There were also scattered mentions of applications, but this is a topic to which I wish to return later.

There seems to be general agreement that the teaching of high school geometry must be modernized, but there is a certain lack of ideas as to how this should be achieved. I recall the detailed debate at the 1958 International Congress on this particular topic, and I am under the impression that this problem is still far from settled.

For example, the School Mathematics Study Group in the United States wrote single textbooks for each of six years for junior high school and high school mathematics. However, in the case of the tenth year, there are already two different versions of geometry available, and there may very well be a third version. This is a clear-cut indication of the lack of agreement amongst leading mathematicians in the United States as to the " right " way of teaching geometry.

The most constructive suggestions on this topic seem to be contained in the report from Germany, and I refer the reader to the excellent bibliography contained in the appendix. I share the astonishment expressed by the German reporter that high school geometry has remained so terribly tradition-bound, even in the face of many changes in the teaching of algebra, and the introduction of more advanced topics. We must choose between a 2,000-year-old tradition of teaching synthetic geometry in the manner of Euclid, or of destroying the " purity " of geometry by the introduction of algebraic ideas. Of course, Felix Klein established a very important trend in Germany, which spread throughout the world, to attempt to build a classification of geometries by means of the transformations which leave certain geometric properties invariant. This points to the importance of the study of geometric transformations, even within high school geometry. There is also an increasing tendency to introduce metric ideas early into synthetic geometry and in many countries even an introduction to analytic geometry is part of the first year's geometry course.

The introduction of vectors is quite generally advocated. In Germany vectors are introduced in the context of metric (as opposed to affine) geometry. However, this does not mean that vectors are tied to analytic geometry, since vector methods are used as a substitute for the introduction of a coordinate system. This approach is particularly useful in bringing out the analogy between the geometries of two, three, and more dimensions.

A conference sponsored by ICMI at Aarhus, in Denmark, in 1960, advocated the development of a " pure " vector geometry, in which affine geometry is built up in terms of vector ideas. While the concept of vectors free of coordinate systems may be somewhat more difficult for the beginning student to understand, many geometric proofs actually become much simpler if vectors are treated as coordinate-free. For example, this is by far the easiest way to prove that the medians of a triangle meet at one point and divide each other in a 2:1 ratio.

While there are still many advocates of treating a full axiomatic system of Euclidean geometry purely synthetically, it is becoming increasingly clear that one must either " cheat " or demand more of the student than can be expected of him in his high school years. Even Euclid's original axiom system is a great deal more complex than is ideal for the high school student's first introduction to axiomatic mathematics. In addition, it is well known that Euclid in many places substituted intuition or the drawing of a diagram for mathematical rigor. Indeed, many of Euclid's propositions do not follow from his axioms. While several outstandingly fine axiom systems have been constructed that make Euclidean synthetic geometry rigorous (notably the system by Hilbert), these require a degree of mathematical maturity not to be expected of the secondary school student.

The report from Israel feels that the axiomatic treatment of geometry in high school is as unrealistic as using Peano's postulates in elementary school. The report from the United States, in contrast, advocates that certain *segments* of Euclidean geometry be taught rigorously, to give the student experience in proving theorems from axioms, but that the gaps in between be filled in by a more intuitive presentation, in which the emphasis should be in teaching students the " facts of geometry ". An alternative to this is the much heavier reliance on the properties of real numbers to fill in gaps in Euclid's axiom system.

Three reports advocated the inclusion of non-Euclidean geometry as part of the first treatment of Euclid. The argument for this is similar to the argument for teaching algebraic systems to improve the students' understanding of number systems. That is, if the student is forced to reason in a geometric framework other than the one he is used to, he is more likely to understand the power of the deductive system and to appreciate proofs he has seen in Euclidean geometry. I should like to add a plea that, even in courses where no actual non-Euclidean geometry is taught, the student should at least be informed that such geometries do exist, and perhaps a day or two be spent discussing them. It seems to me to be a major cultural crime of most mathematical educational systems that 130 years after the invention on non-Euclidean geometry, most students (and many teachers) are not aware of the possibility of a non-Euclidean geometry. Indeed, the statement that our universe is only approximately Euclidean, according to relativity theory — it may both in the small and the large be non-Euclidean — comes as a great shock to many pedagogues.

A frequently mentioned topic is a brief study of relations in general, with special emphasis on equivalence relations and order relations. The justification for such fundamental concepts is the same as for a brief study of sets and of symbolic logic; once these concepts are introduced, they can be used again and again to clarify later topics.

Three reports suggested the inclusion of a systematic study of cardinal numbers. I must say that this suggestion both delights me and surprises me. It delights me in that I have always been critical of university education in the United States, in that most students are supposed to learn the facts about infinite cardinals entirely on their own, since these topics are rarely explicitly taught in courses. The suggestion surprised me because I had felt that this topic was too difficult for high school curricula. If various countries succeed in this experiment, I think it would be most useful if the results were widely publicized.

Suggestions of a brief introduction to topology are contained in four reports. The French report proposes that an intuitive notion of neighborhoods be given to students and on this one should base the concepts of the convergence of a sequence (or the failure of convergence) and that these ideas should be used to lead in a natural way to the concepts of limits and continuity. These can in turn be used to explain such geometric ideas as that of a tangent or of an asymptote. Germany and Israel make similar suggestions.

A more ambitious program is outlined in the Polish report. The proposal is that most of the treatment be restricted to the topology of Euclidean space of one, two, and three dimensions. Starting with these well-known spaces, the concept of a metric space should be developed, and, in turn, illustrated on such examples as n-dimensional space, the space of continuous functions, and Hilbert space. The Polish program would start with the same concepts as mentioned above from the French report. However, by limiting itself to more concrete examples, it proposes to go considerably further. Such general concepts as connectedness, boundary, homeomorphism, and continuous mappings would be discussed. More concretely, it is suggested that discussions without proofs should be given of the Jordan-curve theorem, classification of polyhedral surfaces, and some examples of non-orientability of surfaces. The unit would terminate with a discussion of Euler's theorem.

Young reporter would like to add his support to this suggestion, even though it may sound quite extreme. While these topics may be too difficult for the average high school student, I know from personal experience that the really bright student, in his last year of high school, is fascinated by elementary topological ideas. Such a unit should be entirely practical as long as it is closely tied to concrete examples familiar to the student.

Most of the reports contained frequent mentions of traditional topics whose teaching would be improved by the adoption of a more modern point of view. As one example, I shall use a unit discussed in the report from the United States. This is the treatment of equations, simultaneous equations and inequalities. An equation or inequality is treated as an " open sentence ". That is, it is a mathematical assertion which in itself is neither true nor false, but becomes true or false when its variables are replaced by names of numbers or points (or more abstract objects, in advanced subjects). Therefore, the solution of an equation is the search for the set for which the assertion is true. This set is commonly referred to as the " truth set " or the " solution set ".

Thinking of solutions of equations as sets has the advantage that a student is more likely to think of the possibilities of the solution having more than one element in it or, for that matter, being the empty set. Simultaneous equations may be thought of as conjunctions of several open sentences; hence their solution consists of the intersection of the individual truth sets. This point of view makes it much easier to explain the usual algorithms for solving of simultaneous equations. The attempt in any such algorithm is to replace a set of sentences by an equivalent set, i.e., one having the same truth set, but the latter being of a form in which the nature of the solutions is obvious. The approach also has the advantage that equations and inequalities may be treated in exactly the same manner. The graphing of equations and inequalities, then, simply becomes a matter of graphical representation of truth sets. In this case, the meaning of " intersection " of solution sets becomes particularly clear.

4. Applications of mathematics

It is painfully clear, in reading the 21 national reports, that relatively little attention has been given by our reformers to