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ON THE EXACTNESS OF INTERLOCKING SEQUENCES

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A sequence of abelian groups[1], and homomorphisms

$$\ldots\ldots A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \ldots\ldots \qquad (1)$$

s said to be exact if, for each n, Ker $d_n = \text{Im } d_{n+1}$. If we are presented with such a sequence, the verification of this property usually takes two steps: first, $d_n d_{n+1} = 0$ (i.e. Im $d_{n+1} \subseteq$ Ker d_n), then Ker $d_n \subseteq$ Im d_{n+1}. When we have a more-or-less complicated array of groups and homomorphisms, it is frequently the case that properties of exactness at various points are interrelated. We propose in this note to study the exact degree of interrelation for a simple diagram which often seems to turn up.

Let us start by considering a triple of spaces $X \subset Y \subset Z$. Each pair of spaces from this triple has a homology exact sequence ([1], p. 11); for example, that of $X \subset Y$ is

$$\begin{aligned}\ldots\ldots H_{n+1}(Y, X) \to H_n(X) \to H_n(Y)&\\ \to H_n(Y, X) \to H_{n-1}(X) \ldots\ldots& \qquad (2)\end{aligned}$$

In addition, there is the so-called exact sequence of the triple ([1], p. 25), which runs

$$\begin{aligned}\ldots\ldots H_{n+1}(Z, Y) \to H_n(Y, X) \to H_n(Z, X)&\\ \to H_n(Z, Y) \to H_{n-1}(Y, X) \ldots\ldots& \qquad (3)\end{aligned}$$

We find in [1] a proof (pp. 25-28) that the exactness of (3) follows from the exactness of the sequences for pairs, and from certain commutative diagrams. This is the first result of the kind we

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[1] All the ideas below are equally applicable in any abelian category.

are interested in. Let us observe that the four exact sequences above (of three pairs and a triple) can be fitted into a single diagram as follows (this is, I believe, due to M. A. Kervaire).

$$\begin{tikzcd}[column sep=small]
 & H_{n+1}(Z,Y) \arrow[dr] \arrow[rr, bend left] & & H_n(Y,X) \arrow[dr] \arrow[rr, bend left] & & H_{n-1}(X) \\
H_{n+1}(Z,X) \arrow[ur] \arrow[dr] & & H_n(Y) \arrow[ur] \arrow[dr] & & H_n(Z,X) \arrow[ur] \arrow[dr] & \\
 & H_n(X) \arrow[ur] \arrow[rr, bend right] & & H_n(Z) \arrow[ur] \arrow[rr, bend right] & & H_n(Z,Y)
\end{tikzcd} \qquad (4)$$

This picture may be described as follows. Draw the graphs of the four curves $y = \pm \sin x$, $y = \pm \cos x$. At each point where two curves meet, we put a group; the parts of the curves in between represent homomorphisms. The diagram is commutative, and each of the four curves represents an exact sequence.

We shall use the following notation:

$$\begin{tikzcd}[column sep=tiny]
 & B_{n+1} \arrow[dr] \arrow[rr, bend left] & & E_{n+1} \arrow[dr] \arrow[rr, bend left] & & C_n \arrow[dr] \arrow[rr, bend left] & & F_n \arrow[dr] & \\
A_{n+1} \arrow[ur] \arrow[dr] & & D_{n+1} \arrow[ur] \arrow[dr] & & A_n \arrow[ur] \arrow[dr] & & D_n \arrow[ur] \arrow[dr] & & A_{n-1} \\
 & C_{n+1} \arrow[ur] \arrow[rr, bend right] & & F_{n+1} \arrow[ur] \arrow[rr, bend right] & & B_n \arrow[ur] \arrow[rr, bend right] & & E_n \arrow[ur] &
\end{tikzcd} \qquad (5)$$

This diagram arises in other ways: for example, since the diagram is self-dual, it applies also to cohomology. We may also replace homology in (4) by homotopy. If, in particular, $X \subset Y \subset Z$ are Lie subgroups of a Lie group, we can also interpret $\Pi_n(Y, X)$ as $\Pi_n(Y/X)$, and so have a sequence of absolute homotopy groups of certain homogeneous spaces. The diagram also appears in cobordism theory.

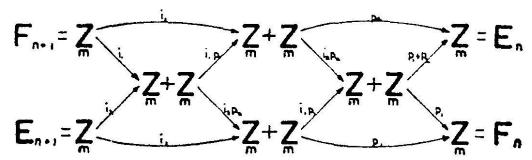
One further example is of interest: let $(W; X, Y)$ be a proper triad ([1], p. 34), with $Z = X \cap Y$, i.e. $H_r(Y, Z) \cong H_r(W, X)$ and $H_r(X, Z) \cong H_r(W, Y)$ by inclusion. Then the four exact sequences (2) form the diagram:

$$\begin{tikzcd}[column sep=tiny]
 & H_{r+1}(W,Y)=H_{r+1}(X,Z) \arrow[dr] \arrow[rr, bend left] & & H_r(Y) \arrow[dr] \arrow[rr, bend left] & & H_r(W,X)=H_r(Y,Z) \arrow[dr] & \\
H_{r+1}(W) \arrow[ur] \arrow[dr] & & H_r(Z) \arrow[ur] \arrow[dr] & & H_r(W) \arrow[ur] \arrow[dr] & & H_{r-1}(Z) \\
 & H_{r+1}(W,X)=H_{r+1}(Y,Z) \arrow[ur] \arrow[rr, bend right] & & H_r(X) \arrow[ur] \arrow[rr, bend right] & & H_r(W,Y)=H_r(X,Z) \arrow[ur] &
\end{tikzcd}$$

This suggests that the diagram is related to Mayer-Vietoris sequences ([1], p. 39).

Our results about exactness of the diagram are as follows (proofs in this paper will be trivial, and all left to the reader).

(1) *Suppose that the diagram (5) is commutative, and that the three sequences (A, B, E), (A, C, F), (B, D, F) are exact. Then the sequence (C, D, E) is exact at C and at E, and if either of* Im $(C_n \to D_n)$, *Ker* $(D_n \to E_n)$ *contains the other, the two are equal. (The sequence is not necessarily exact at D.)* This already saves some labour if exactness of four such sequences is to be verified: the 24 ' parts of exactness ' (2 at each term of each sequence) are reduced to 19. For our counterexample, we have



(7)

in which all unwritten terms vanish. Now for the Mayer-Vietoris sequence.

(2) *If (5) is a commutative exact diagram, the following is exact*

$$A_n \to B_n \oplus C_n \to D_n \to A_{n-1} \,. \qquad \text{(Proof, [1] pp. 39-41)} \qquad (8)$$

Here, all homomorphisms are the obvious ones, except that we must change a sign, say of $C_n \to D_n$. Note that the statement that the composite $A_n \to D_n$ vanishes gives commutativity of a square. Also note that in (5), we can give 8 sequences of homomorphisms (represented by straight lines), and define the other 4 as compositions.

(3) *Given 8 sequences of homomorphisms, as in (5), we obtain a commutative exact diagram if and only if each of the short sequences*

$$A_n \to B_n \oplus C_n \to D_n \,, \quad D_n \to E_n \oplus F_n \to A_{n-1} \,, \quad B_n \to D_n \to F_n \,,$$
$$F_{n+1} \to A_n \to C_n \,, \quad C_n \to D_n \to E_n \,, \quad E_{n+1} \to A_n \to B_n \textit{ is exact.}$$

This is a much more efficient theorem than (1): we have reduced the 24 parts to 10 (more precisely, 26 to 12).

(4) *An equivalent condition is that the two Mayer-Vietoris sequences are exact, and the compositions* $B_n \to D_n \to F_n$, *etc., are zero.*

We now consider two situations where more complicated diagrams arise: these are also interesting, but presumably less important. First consider $(n-1)$ spaces linearly ordered by inclusion: write $\emptyset = X_0 \subset X_1 \subset \ldots \subset X_{n-1}$. We shall find the diagram containing all sequences (2) and (3). We set $G_{ij} = H^0(X_i, X_j)$. Thus for inclusion mappings, i and j are increased: these are all composites of $G_{ij} \to G_{i+1, j}$ and $G_{ij} \to G_{i, j+1}$ for various i, j. To accommodate

$$\delta^* : H^\circ(X_i, X_j) \to H^1(X_j, X_k) ,$$

we factorise into irreducible maps

$$H^\circ(X_i, X_j) \to H^\circ(X_{i+1}, X_j) \to \ldots \to H^\circ(X_{n-1}, X_j) \to H^1(X_j, X_0) \to$$
$$\to H^1(X_j, X_k) .$$

This suggests that we set $G_{nj} = H^1(X_j, X_0)$ and generally $H^1(X_j, X_k) = G_{n+k, j}$. Eventually, we put

$$H^{2r}(X_i, X_j) = G_{i+rn, j+rn}, \quad H^{2r+1}(X_i, X_j) = G_{j+n(r+1), i+rn} ,$$

for $0 \leqslant j < i < n$, $r \in Z$. So G_{uv} is defined whenever $u, v \in Z$, $u - n < v < u$. All maps are composites of maps $G_{u,v} \xrightarrow{\beta} G_{u, v+1}$ and $G_{u,v} \xrightarrow{\alpha} G_{u+1, v}$ and all the squares commute. The exact sequence of the triple (X_i, X_j, X_k) with $i > j > k$ now appears as

$$\begin{aligned} G_{j+rn, k+rn} \xrightarrow{\alpha} G_{i+rn, k+rn} \xrightarrow{\beta} G_{i+rn, j+rn} \xrightarrow{\alpha} G_{k+(r+1)n, j+rn} \xrightarrow{\beta} \\ G_{k+(r+1)n, i+rn} \xrightarrow{\alpha} G_{j+(r+1)n, i+rn} \xrightarrow{\beta} G_{j+(r+1)n, k+(r+1)n} . \end{aligned} \tag{9}$$

(3_n) *Given* G_{uv} *for* $u - n < v < u$, *and homomorphisms*

$$G_{u,v} \xrightarrow{\alpha} G_{u+1, v} , \quad G_{u,v} \xrightarrow{\beta} G_{u, v+1} ,$$

these form a commutative diagram, with sequences (9) exact, if and only if all the sequences below are exact.

$$G_{u,v} \xrightarrow{(\alpha,\,\beta)} G_{u+1,\,v} \oplus G_{u,\,v+1} \xrightarrow{(\beta,\,-\alpha)} G_{u+1,\,v+1}$$

$$G_{u+1,\,u} \xrightarrow{\alpha} G_{u+2,\,u} \xrightarrow{\beta} G_{u+2,\,u+1}$$

$$G_{u,\,u-n+1} \xrightarrow{\beta} G_{u,\,u-n+2} \xrightarrow{\alpha} G_{u+1,\,u-n+2}\ .$$

Secondly let us consider all the homotopy groups and exact sequences we can obtain from a triad—or, more precisely, from a commutative diagram

$$\begin{tikzcd}[row sep=small, column sep=small] & B \arrow[dr] & \\ D \arrow[ur] \arrow[dr] & & A \\ & C \arrow[ur] & \end{tikzcd}$$

of maps which need not be inclusions. We have the homotopy groups of 4 spaces and of 5 pairs. There are also some quadruples (in the sense of Eckmann-Hilton). Let us observe that for any quadruple which has a cross-homomorphism making the diagram

$$\begin{tikzcd}[row sep=small, column sep=small] & X \arrow[dr] \arrow[dd] & \\ W \arrow[ur] \arrow[dr] & & Z \\ & Y \arrow[ur] & \end{tikzcd}$$

commutative, the n^{th} homotopy group splits as $\Pi_n(Y, Z) \oplus \Pi_{n-1}(W, X)$: one exact sequence is trivial, the other of Mayer-Vietoris type. If we exclude such quadruples, there remain only 3:

$$\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr, phantom, "\Phi"] & B \arrow[d] \\ C \arrow[r] & A \end{tikzcd} \qquad \begin{tikzcd} D \arrow[r] \arrow[d] \arrow[dr, phantom, "X"] & B \arrow[d] \\ C \arrow[r] & A \end{tikzcd} \qquad \begin{tikzcd} D \arrow[r] \arrow[d] \arrow[dr, phantom, "\Psi"] & B \arrow[d] \\ C \arrow[r] & x \end{tikzcd}$$

where x denotes a point (or base-point). Clearly we have 5 exact sequences of pairs, 2 of triples, 6 of quadruples. There are also 2 of octuples, induced by the obvious maps $\Phi \to X$ and $X \to \Psi$. But, for example, the first octuple has the lower quadruple trivial, hence the same homotopy groups (with dimension shift) as

$$\begin{tikzcd} x \arrow[rrr] \arrow[ddd] \arrow[dr] & & & B \arrow[ddd] \arrow[dl, equal] \\ & D \arrow[r] \arrow[d] & B \arrow[d] & \\ & C \arrow[r] \arrow[dl, equal] & A \arrow[dr, equal] & \\ C \arrow[rrr] & & & A \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r] \arrow[d] & B \arrow[d, equal] \\ D \arrow[r] & B \end{tikzcd}$$

here a pair is trivial, so we have the same homotopy groups (with another dimension shift) as D. The other octuple similarly reduces to A.

Thus we have 12 sequences of groups, which lie in 15 exact sequences; these we write as

$$\begin{tikzcd}[column sep=small]
 & B_1 \arrow[dr] & & D_3 \arrow[dr] & & F_1 \arrow[dr] & & B_3 \arrow[dr] & & D_1 \arrow[dr] & & F_3 \arrow[dr] & & B_1 \arrow[dr] & \\
A \arrow[ur] \arrow[r] \arrow[dr] & B_2 \arrow[r] & C \arrow[ur] \arrow[r] \arrow[dr] & D_2 \arrow[r] & E \arrow[ur] \arrow[r] \arrow[dr] & F_2 \arrow[r] & A \arrow[ur] \arrow[r] \arrow[dr] & B_2 \arrow[r] & C \arrow[ur] \arrow[r] \arrow[dr] & D_2 \arrow[r] & E \arrow[ur] \arrow[r] \arrow[dr] & F_2 \arrow[r] & A \arrow[ur] \arrow[r] \arrow[dr] & B_2 \arrow[r] & C \\
 & B_3 \arrow[ur] & & D_1 \arrow[ur] & & F_3 \arrow[ur] & & B_1 \arrow[ur] & & D_3 \arrow[ur] & & F_1 \arrow[ur] & & B_3 \arrow[ur] &
\end{tikzcd} \tag{10}$$

where

$$B_i \to C \to D_i\,, \quad D_i \to E \to F_i\,, \quad F_i \to A \to B_i \ (i = 1, 2, 3) \ \text{and}$$
$$B_i \to D_j \to F_k$$

$((i, j, k)$ a permutation of $(1, 2, 3))$ are the 15 sequences. Here we have set

$$\begin{aligned}
A &= \Pi_{n+1}(\Psi)\,, & B_1 &= \Pi_n(A)\,, & B_2 &= \Pi_n(B, D)\,,\\
B_3 &= \Pi_n(C, D)\,, & C &= \Pi_n(A, D)\,, & D_1 &= \Pi_{n-1}(D)\,,\\
D_2 &= \Pi_n(A, B)\,, & D_3 &= \Pi_n(A, C)\,, & E &= \Pi_n(\Phi)\,,\\
F_1 &= \Pi_n(X)\,, & F_2 &= \Pi_{n-1}(C)\,, & F_3 &= \Pi_{n-1}(B)\,.
\end{aligned}$$

This diagram also contains an immense number of diagrams (4), each with two Mayer-Vietoris sequences (8): we shall not go into any more details.

REFERENCES

[1] S. Eilenberg and N. E. Steenrod, *Foundations of Algebraic Topology*. Princeton, 1952.