3. BOUNDEDNESS THEOREMS II

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(b) $\int | a(s) | s^{\alpha} ds < \infty$,

(c) $0 < g(v) \leqq K$ for all v;

then the derivative of any solution of (1.1) has a limit.

Proof. Proceeding as in the above proof, we obtain instead of (3.2) the following estimate:

$$\frac{| u(t) |}{t} \leqq (| u(t_0) | + | u'(t_0) |) + \int_{t_0}^{t} s^{\alpha} KM \, | a(s) | \, h\left(\frac{| u(s) |}{s}\right) ds \, ,$$

from which we conclude from a result of Bihari [14] that

$$\frac{| u(t) |}{t} \leqq H^{-1}\left(H(| u(t_0) | + | u'(t_0) |) + KM \int_{t_0}^{t} | a(s) | \, s^{\alpha} ds\right)$$

which is bounded for t on account of assumption *(a)*. The remaining proof follows verbatim that of Theorem 3.

3. Boundedness Theorems II

Theorem 5. Suppose that assumptions A_1, A_2, A_3 and A_4 hold and in addition that

(i) $a(t) > 0, \quad a'(t) \geqq 0, \quad$ for $\quad t \geqq T$,

(ii) $\frac{d}{dt}\left(\frac{b}{a}\right) \leqq \beta(t)\left(1 + \frac{b}{a}\right)$, with $\int^{\infty} \beta(s) \, ds < \infty$

and

$$\left(1 + \frac{b}{a}\right) \geqq \varepsilon > 0 \, ;$$

then every solution of (1.1) with $(a(t) + b(t))$ replacing $a(t)$ is bounded.

Proof. Make the following substitution for the independent variable, $x = \int^{t} \sqrt{a(s)} \, ds$ which tends to infinity as $t \to \infty$, and obtain instead of (1.1) its transformed equation:

$$\frac{d^2 u}{dx^2} + \frac{1}{2}\left(\frac{a}{a^{3/2}}\right)\frac{du}{dx} + \left(1 + \frac{b}{a}\right) f(u) \, g(u') = 0 \qquad (3.1)$$

where " dot " denotes differentiation with respect to t. Now write equation in its system form, letting $y_1 = u$:

$$\begin{cases} \dfrac{d y_1}{dx} = y_2 \\ \dfrac{d y_2}{dx} = -\dfrac{1}{2}\left(\dfrac{a}{a^{3/2}}\right) y_2 - \left(1 + \dfrac{b}{a}\right) f(y_1) g(\sqrt{a}\, y_2) . \end{cases} \tag{3.2}$$

Define for (3.2) the following function:

$$V(x, y_1, y_2) = \left(1 + \frac{b}{a}\right) \int^{y_1} f(s)\, ds + \int^{y_2} \frac{s\, ds}{g(\sqrt{a}\, s)} ,$$

and observe:

$$\begin{aligned} \frac{dV}{dx} &\leqq \frac{\beta(t)}{\sqrt{a(t)}} \left(1 + \frac{b}{a}\right) \int^{y_1} f(s)\, ds - \frac{1}{2} \frac{a}{a^{3/2}} y_2{}^2 \\ &\leqq \frac{\beta(t)}{\sqrt{a(t)}} V . \end{aligned}$$

Hence we have

$$V(x, y_1, y_2) \leqq V\big(x(T),\ y_1(x(T)),\ y_2(x(T)\big) \exp \int_T^t \beta(s)\, ds$$

which is finite. From (ii) we note that $V \to \infty$ as $y_1 \to \infty$ and $V > 0$ if $y_1^2 + y_2^2 \neq 0$. Thus, every solution of (1.1) is bounded.

Corollary. Suppose in addition to the hypothesis of Theorem 5 that assumption A_5 also holds and that $\lim\limits_{t \to \infty} a(t) = a_1 < \infty$, then every solution of (1.1) and its derivative are bounded.

From the above result we may conclude for example that all solutions of the following equation:

$$u''(t) + (c_1 t^\alpha + c_2 t^\beta) u^\lambda(t) \big(1 + \exp u'(t) \sin u'(t)\big) = 0$$

are bounded for all c_1, $c_2 > 0$, $\alpha > \beta \geqq 0$, and $\lambda > 0$.

We now consider the following inhomogeneous equation:

$$u''(t) + a(t) f(u) g(u') = h(t, u, u') \tag{3.3}$$

and assume that $| u' h(t, u, u') | \leqq \gamma(t) g(u')$ where $\int^{\infty} \gamma(s)\, ds < \infty$.

Theorem 6. Suppose that assumptions A_1, A_2, A_3 and A_4 hold and in addition that $a(t) > 0$ and $a'(t) \geqq 0$ for $t \geqq T$; then all solutions of (3.3) are bounded.

Proof. Integrate (3.3) in the following manner:

$$G(u'(t)) - G(u'(t_0)) + a(t) F(u(t)) - a(t_0) F(u(t_0))$$
$$= \int_{t_0}^{t} a'(s) F(u(s)) ds + \int_{t_0}^{t} \frac{h(t, u, u') u'(s) ds}{g(u')} \qquad (3.4)$$

where $G(v) = \int_0^{\gamma} \frac{s\, ds}{g(s)}$ and $F(u) = \int_0^{u} f(s)\, ds$. Taking absolute values and noting that $G(v) \geqq 0$ and $F(u) \geqq 0$, we obtain

$$a(t) F(u(t)) \leqq c_0 + c_1 + \int_{t_0}^{t} a'(s) F(u(s)) ds \qquad (3.5)$$

where $c_0 = G(u'(t_0)) + a(t_0) F(u(t_0\, O)$ and $c_1 = \int_{t_0}^{\infty} \gamma(s)\, ds$ are non-negative constants. From (3.5) and A_4 it is now clear that every solution of (3.3) are bounded (cf. [1]).

Corollary. In addition to the hypothesis of Theorem 6, suppose that assumption A_5 also holds and that $\lim_{t \to \infty} a(t) = k > 0$; then all solutions of (3.3) and their derivatives are bounded.

We note that by setting $h(t, u, u') \equiv 0$, the above result again reduces to Theorem 1 and its corollary. Other comparison theorems may be formulated in a similar way as Theorem 6 by extending the corresponding result for the homogeneous equation. Since the procedure is clear, the statements and proofs of these results will be omitted.

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