

## 3.1. Local theory.

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$\exp(t\xi)$  must leave  $x$  invariant, and therefore  $\xi$  must vanish at  $x$  (this was the case in the preceding example).

*The Zariski tangent space.* The Zariski tangent space at a point  $x$  of an analytic space  $X$  is the dual over  $\mathbb{C}$  of  $\mathfrak{M}_x/\mathfrak{M}_x^2$ ; here  $\mathfrak{M}_x$  denotes as usual the maximal ideal of  $\mathcal{O}_{X,x}$ . If  $X$  is defined by the ideal  $\mathcal{I} \subset \mathcal{O}_U$ ,  $U$  an open set in  $\mathbb{C}^n$ , the tangent space may be identified with the linear variety defined by the linear parts of all germs  $\in \mathcal{I}_x$ .

The Zariski tangent space of  $X_{red}$  may be strictly contained in that of  $X$ . For instance, if  $X$  is a double point,  $\mathfrak{M}_x/\mathfrak{M}_x^2$  has dimension 1 over  $\mathbb{C}$  whereas  $\mathfrak{M}_x/\mathfrak{M}_x^2 = \{0\}$  for  $X_{red}$ , the corresponding simple point.

*The tangent cone.* The tangent cone at a point  $x$  of a local model  $(X, \mathcal{O}_X)$  is the algebraic variety (with nilpotents, in general) defined by the ideal generated by the first non-vanishing homogeneous parts of the elements in  $\mathcal{I}_x$ ,  $\mathcal{I}$  being the ideal defining  $X$ . Since the Zariski tangent space is defined, in the local model, by the ideal spanned by the first-degree parts of the elements of  $\mathcal{I}_x$  it is clear that it contains, and in general strictly, the tangent cone. If  $\xi$  is a vector field,  $\xi(x)$  belongs to the reduced tangent cone at  $x$ , but since the possible values of  $\xi(x)$  form a linear space, it is in general not equal to the whole cone.

*Example 3.* Let, again,  $X$  be the analytic subspace of  $\mathbb{C}^2$  defined by the ideal  $(x^3 - y^2)$ . Then, as noted before,  $\xi(x) = 0$  for all possible vector fields; the tangent cone is the algebraic variety defined by the ideal  $(y^2)$ , and the reduced tangent cone is the variety  $y = 0$ ; finally, the Zariski tangent space is the whole space  $\mathbb{C}^2$ , for  $x^3 - y^2$  contains no linear terms.

## CHAPTER 3.

### FINITE MORPHISMS

#### 3. 1. Local theory.

As elsewhere in these notes, we denote by  $\mathbb{C}\{x_1, \dots, x_n\}$  the ring of convergent power series in  $n$  variables  $x_1, \dots, x_n$ . First, we recall the so-called "Weierstrass preparation theorem".

*Theorem 3.1.1.* (Späh, Rückert). Given  $\Phi \in \mathbb{C}\{x_1, \dots, x_n\}$ , with  $\Phi(0, \dots, 0, x_n) = x_n^p + (\text{higher order terms})$ , any  $f \in \mathbb{C}\{x_1, \dots, x_n\}$  can

be written

$$f = \Phi Q + \sum_{i=0}^{p-1} x_n^i a_i$$

with  $Q \in \mathbb{C} \{x_1, \dots, x_n\}$ ,  $a_i \in \mathbb{C} \{x_1, \dots, x_{n-1}\}$

This representation is unique.

*Corollary 3.1.2.* (Weierstrass). Given  $\Phi$  as in the preceding theorem, there exist  $u \in \mathbb{C} \{x_1, \dots, x_n\}$ , with  $u(0) \neq 0$  and  $a_i \in \mathbb{C} \{x_1, \dots, x_{n-1}\}$ , with  $a_i(0) = 0$  such that

$$\Phi u = x_n^p + \sum_{i=0}^{p-1} a_i x_n^i$$

$u$  and  $(a_i)$  are unique.

The corollary results easily from the theorem, when applied to  $f = x_n^p$ . For the proof of theorem 3.1.1., see e.g. [5] or [9]. We recall also that theorem 3.1.1. implies the facts that  $\mathbb{C} \{x_1, \dots, x_n\}$  is noetherian, and is a unique factorisation domain.

*Definition 3.1.3.* An analytic algebra (we shall say also “analytic ring”) is a quotient  $\mathbb{C} \{x_1, \dots, x_n\} / \mathcal{I}$ , where  $\mathcal{I}$  is a non trivial ideal (i.e.  $\mathcal{I} \neq \mathbb{C} \{x_1, \dots, x_n\}$ ). An analytic algebra  $A$  is clearly a local  $\mathbb{C}$ -algebra ; we denote by  $\mathfrak{M}(A)$  its maximal ideal ; we have  $A/\mathfrak{M}(A) \simeq \mathbb{C}$ .

An analytic algebra, being a quotient of a noetherian ring, is a noetherian ring, and therefore is separated in the Krull topology (see appendix).

If  $A$  and  $B$  are two analytic algebras, and  $f: A \rightarrow B$  a homomorphism (with  $f(1) = 1$ ), we recall that  $f$  is automatically local and therefore continuous in the Krull topology (see § 1.2.). If  $E$  is a  $B$ -module (unitary), then the map  $A \times E \rightarrow E$  defined by  $(a, e) \rightarrow f(a) e$  makes  $E$  an  $A$ -module ; for simplicity, we write  $f(a) e = a e$ .

We can now state the preparation theorem, in the general form :

*Theorem 3.1.3.* Let  $A$  and  $B$  be analytic algebras,  $f$  a homomorphism  $A \rightarrow B$ , and  $E$  a finite  $B$ -module. Then  $E$  is finite over  $A$  if and only if  $E/\mathfrak{M}(A)E$  is finite over  $A/\mathfrak{M}(A) \simeq \mathbb{C}$  (by “finite over  $A$ ” we mean “finitely generated as an  $A$ -module”).

This theorem can be precised as follows :

*Corollary 3.1.4.* Given  $A, B, f, E$  as above, suppose that the images of  $e_1, \dots, e_p$  in  $E/\mathfrak{M}(A)E$  generate that module over  $\mathbb{C}$ ; then  $e_1, \dots, e_p$  generate  $E$  over  $A$ .

*Proof of the corollary, admitting the theorem.* Let  $F$  be the sub- $A$ -module of  $E$  spanned by  $e_1, \dots, e_p$ ; then, by hypothesis, we have  $E = F + \mathfrak{M}(A)E$ . On the other hand, using the theorem, we know that  $E$  is finite over  $A$ ; we can therefore apply Nakayama's lemme (see Appendix), which proves the corollary.

The existence part of theorem 3.1.1. is a special case of the preceding result. For, we take  $A = \mathbb{C}\{x_1, \dots, x_{n-1}\}$ ,  $B = \mathbb{C}\{x_1, \dots, x_n\}$  and  $f$  the natural injection (or, in a more sophisticated language,  $f = \pi^*$  where  $\pi$  is the projection  $\mathbb{C}^n \rightarrow \mathbb{C}^{n-1}$  which "forgets the last coordinate"); choose now  $\Phi$  as in theorem 3.1.1. and  $E = B/(\Phi)$ . Then  $E/\mathfrak{M}(A)E$  is isomorphic to  $\mathbb{C}\{x_n\}/(\Phi(0, \dots, 0, x_n)) = \mathbb{C}\{x_n\}/(x_n^p)$ , which is generated over  $\mathbb{C}$  by the classes of  $1, x_n, \dots, x_n^{p-1}$ . Therefore, the corollary 3.1.4. shows that the classes of  $1, x_n, \dots, x_n^{p-1}$  in  $E$  generates  $E$  over  $A$ , which is the existence part of theorem 3.1.1.

A direct proof of theorem 3.1.3. in a slightly less general case ( $E = B$ ) can be found in [6] (the general case could be easily deduced of it). We shall follow here another method, used by Mather [7] in the  $\mathbb{C}^\infty$ -case, and deduce theorem 3.1.3. from theorem 3.1.1. We proceed in three steps.

*Step 1.*  $A = \mathbb{C}\{x_1, \dots, x_{n-1}\}$ ,  $B = \mathbb{C}\{x_1, \dots, x_n\}$ ,  $f = \pi^*$ , the natural injection  $A \rightarrow B$ . As in the theorem,  $E$  is a finite  $B$ -module such that  $E/\mathfrak{M}(A)E$  is finite over  $\mathbb{C}$ .

We first prove the existence of a finite number of elements  $e_1, \dots, e_p$  in  $E$  such that any  $e \in E$  can be written  $e = \sum b_i e_i$ , with  $b_i \in f(A) + \mathfrak{M}(A)B$ . To this end, let  $\varepsilon_1, \dots, \varepsilon_q$  generate  $E$  over  $B$ , and let  $\eta_1, \dots, \eta_r$  be members of  $E$  such that their classes  $\bar{\eta}_1, \dots, \bar{\eta}_r$  modulo  $\mathfrak{M}(A)E$  generate  $E/\mathfrak{M}(A)E$  over  $\mathbb{C}$ . Thus, for any  $e \in E$ , we have, for suitable  $\gamma_i \in \mathbb{C}$

$$e - \sum \gamma_i \eta_i \in \mathfrak{M}(A)E$$

and therefore

$$e - \sum \gamma_i \eta_i = \sum b_j \varepsilon_j, \quad b_j \in \mathfrak{M}(A)B$$

and it suffices to take  $p = q + r$ ,  $(e_1, \dots, e_p) = (\eta_1, \dots, \eta_r, \varepsilon_1, \dots, \varepsilon_q)$

Therefore, for  $1 \leq i \leq p$ , we have

$$x_n e_i = \sum_j v_{ij} e_j, \quad v_{ij} \in f(A) + \mathfrak{M}(A)B$$

(in other words,  $v_{ij}(0, \dots, 0, x_n)$  is a constant). If we put  $\Phi = \det(x_n \delta_{ij} - v_{ij})$ , we have  $\Phi e_i = 0$   $i = 1, \dots, p$ , then  $\Phi E = 0$ . Therefore  $E$  is a module over  $B/(\Phi)$ , generated e.g. by  $e_1, \dots, e_p$ . But  $\Phi(0, \dots, 0, x_n)$  is a monic polynomial of degree  $p$ , and therefore is not identically zero; by theorem

3.1.1.,  $B/(\Phi)$  is finite over  $A$ , and is generated by  $1, x_n, \dots, x_n^k$  for some  $k \leq p - 1$ . Then  $E$  is finite over  $A$ .

*Step 2.* We suppose that  $A$  and  $B$  are regular analytic rings:

$$A = \mathbf{C}\{x_1, \dots, x_n\}, \quad B = \mathbf{C}\{y_1, \dots, y_m\}$$

and let  $f$  be any homomorphism  $A \rightarrow B$ .

We factorise  $f$  in the following way

$$\begin{array}{ccc} C = \mathbf{C}\{x_1, \dots, x_n, y_1, \dots, y_m\} & & \\ \begin{array}{c} i \nearrow \\ \searrow j \end{array} & & \\ A = \mathbf{C}\{x_1, \dots, x_n\} \xrightarrow{f} B = \mathbf{C}\{y_1, \dots, y_m\} & & \end{array}$$

where  $i$  is the natural injection, and  $\tilde{f}$  is “the map into the graph” defined by

$$\tilde{f}(x_i) = f(x_i), \quad \tilde{f}(y_j) = y_j$$

By our hypothesis,  $E$  is finite over  $B$ ; then,  $\tilde{f}$  being surjective,  $E$  is finite over  $C$ ; the problem is now reduced to one similar to the first case, except that the number of additional variables is  $m$  instead of 1. The proof follows by repeated use of step 1.

*Step 3. General case.* We have now  $A = \mathbf{C}\{x_1, \dots, x_n\}/\mathcal{I}$ ,  $B = \mathbf{C}\{y_1, \dots, y_m\}/\mathcal{J}$ ; and  $E$  is a finite  $B$ -module such that  $E/\mathfrak{M}(A)E$  is finite over  $\mathbf{C}$ .

First, we put  $A' = \mathbf{C}\{x_1, \dots, x_n\}$  and we denote by  $f' : A' \rightarrow B$  the composition of  $f$  and the natural projection  $A' \rightarrow A$ ; it is clear that  $E/\mathfrak{M}(A')E \simeq E/\mathfrak{M}(A)E$ ; therefore, we can replace  $A$  by  $A'$ .

Now, putting  $B' = \mathbf{C}\{y_1, \dots, y_m\}$  and  $\pi$  the natural projection  $B' \rightarrow B$ , we claim that there exists a commutative diagram of homomorphisms

$$\begin{array}{ccc} \tilde{f} \nearrow & B' & \\ A' & \downarrow \pi & \\ f' \searrow & B & \end{array}$$

For, let  $\varphi_1, \dots, \varphi_n \in B'$  be liftings of  $f'(x_1), \dots, f'(x_n)$ ; there exists a unique homomorphism  $\tilde{f} : A' \rightarrow B'$  such that  $\tilde{f}(x_i) = \varphi_i$ ; for any polynomial  $a \in A$ , we have  $\pi \circ \tilde{f}(a) = f'(a)$ ; therefore, for any  $a \in A$  we have  $\pi \circ \tilde{f}(a) - f'(a) \in \bigcap_k \mathfrak{M}^k(B)$ . But  $B$  is noetherian, hence separated in the Krull topology; therefore, we have  $\bigcap_k \mathfrak{M}^k(B) = \{0\}$ , and  $\pi \circ \tilde{f} = f'$ .

Now, we may consider  $E$  as a finite  $B'$ -module, and we are reduced to consider the situation  $(A', B', \tilde{f}, E)$  instead of the given one; but that case was treated in step 2; this ends the proof of the theorem.

*Remarks.* 1. The same proof applies to the real case, and, more generally, to analytic algebras over a complete valuated field.

2. In the  $C^\infty$  case (over  $\mathbf{R}$ ), it is known that the existence part of theorem 3.1.1. is true. Therefore steps 1 and 2 of the preceding proof are applicable, but not step 3 (the lifting  $\tilde{f}$  cannot be constructed a priori, so one has to suppose that such a lifting exists).

### 3.2. Germs of analytic spaces.

This concept will be introduced in terms of categories. As objects, we take triples  $(X, \mathcal{O}_X, x)$  where  $(X, \mathcal{O}_X)$  is an analytic space, and  $x$  a point of  $X$ ; as morphisms of  $(X, \mathcal{O}_X, x)$  into  $(Y, \mathcal{O}_Y, y)$  we take the germs at  $x$  of morphisms of  $(X, \mathcal{O}_X)$  into  $(Y, \mathcal{O}_Y)$ , which map  $x$  into  $y$ . To simplify the notations, we write  $(X, x)$  for  $(X, \mathcal{O}_X, x)$ .

We shall prove some results on the correspondence between analytic rings and germs of analytic spaces.

*Proposition 3.2.1.* To any germ  $(X, x)$  of an analytic space is associated an analytic ring  $\mathcal{O}_{X,x}$ . Every analytic ring is obtained in this way. Every morphism  $(X, x) \rightarrow (Y, y)$  of germs of analytic spaces induces a homomorphism  $\mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$  of analytic rings. Conversely every homomorphism  $B \rightarrow A$  of analytic rings is obtained from a morphism of corresponding germs of analytic spaces; the latter is unique.

*Proof.* If  $(X, x)$  is a germ of analytic spaces,  $\mathcal{O}_{X,x}$  is an analytic ring by definition. Now let  $A = \mathbf{C}\{x_1, \dots, x_n\}/I$  be an analytic ring. We choose generators  $f_1, \dots, f_p$  for  $I$  and take an open neighborhood  $U$  of 0 such that representatives of  $f_1, \dots, f_p$  which are analytic in  $U$  can be found. These generators then define a coherent sheaf  $\mathcal{I}$  of ideals on  $U$  which defines an analytic subspace  $X$  of  $U$  with  $\mathcal{O}_{X,0} = A$ .

If  $f: B \rightarrow A$  is a homomorphism of analytic rings, we shall construct a morphism  $(X, 0) \rightarrow (Y, 0)$  of corresponding germs which induces  $F$ . We may suppose

$$A = \mathbf{C}\{x_1, \dots, x_n\}/(f_1, \dots, f_p), \quad B = \mathbf{C}\{y_1, \dots, y_m\}/(g_1, \dots, g_q);$$

as we have seen in § 1,  $F$  can be lifted into a homomorphism  $F^1: \mathbf{C}\{y_1, \dots, y_m\} \rightarrow \mathbf{C}\{x_1, \dots, x_n\}$ ; we can choose 1) open sets  $U \subset \mathbf{C}^n$ ,  $V \subset \mathbf{C}^m$  with  $0 \in U$ ,  $0 \in V$  2) holomorphic functions  $\bar{f}_1, \dots, \bar{f}_p$  in  $U$  and  $\bar{g}_1, \dots, \bar{g}_q$  in  $V$  such that their germs at 0 are precisely the  $f_i$ 's and the  $g_j$ 's, and 3) an holomorphic mapping  $\Phi: U \rightarrow V$ , with  $\Phi(0) = 0$  such that  $\Phi^*$  induces  $F'$  at the origin.