

4.2. Topology on (X, F) .

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **14 (1968)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

If X is a Stein space, X_{red} is obviously also a Stein space. The converse is also true (see Grauert [2]).

Theorem 4.1.2. (“Theorems A and B” of Cartan-Oka). Let F be an analytic coherent sheaf over a Stein space (X, \mathcal{O}_X) . Then

- 1) For any $x \in X$, $\Gamma(X, F)$ generates F_x over $\mathcal{O}_{X,x}$
- 2) For $p \geq 1$, one has $H^p(X, F) = 0$

This theorem will not be proved here (see f.i. [5] for the reduced case ; the general case is similar). We will need here only the following special case :

Let (X, \mathcal{O}_X) be a closed analytic subspace of a domain of holomorphy $U \subset \mathbf{C}^n$; if F is an analytic coherent sheaf on X , let \tilde{F} be the trivial extension of F to U ; then \tilde{F} is a coherent sheaf of \mathcal{O}_U -modules, and theorems A and B are valid for \tilde{F} : therefore, they are true for F .

4.2. Topology on $\Gamma(X, F)$.

1. Let X be a closed analytic subspace of a domain of holomorphy $U \subset \mathbf{C}^n$; and, with the previous notations, suppose that \tilde{F} admits a *finite presentation* i.e. an exact sequence of sheaves of \mathcal{O}_U -modules

$$\mathcal{O}_U^q \xrightarrow{\alpha} \mathcal{O}_U^p \xrightarrow{\beta} \tilde{F} \rightarrow 0.$$

Applying theorem B to the exact sequences

$$0 \rightarrow \text{Im } \alpha \rightarrow \mathcal{O}_U^p \rightarrow \tilde{F} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \text{Ker } \alpha \rightarrow \mathcal{O}_U^q \rightarrow \text{Im } \alpha \rightarrow 0$$

we get an exact sequence

$$\Gamma(U, \mathcal{O}_U)^q \xrightarrow{\Gamma(U, \alpha)} \Gamma(U, \mathcal{O}_U)^p \xrightarrow{\Gamma(U, \beta)} \Gamma(U, \tilde{F}) \rightarrow 0.$$

The space $\Gamma(U, \mathcal{O}_U)$, with the topology of uniform convergence on compact sets is a Frechet space. And we claim that, for that topology, $\text{Im } \Gamma(U, \alpha)$ is closed. For, if f is adherent to $\text{Im } \Gamma(U, \alpha)$, it results easily from Krull's theorem (see Appendix) that, for $x \in U$, we have $f_x \in \text{Im } (\alpha_x)$, hence $f \in \Gamma(U, \text{Im } \alpha)$; but, according to theorem B, the mapping $\Gamma(U, \mathcal{O}_U)^q \rightarrow \Gamma(U, \text{Im } \alpha)$ is surjective.

Now, with the quotient topology, $\Gamma(X, F) \simeq \Gamma(U, \tilde{F}) \simeq \Gamma(U, \mathcal{O}_U)/\text{Im } \Gamma(U, \alpha)$ is a Frechet space. This topology does not depend on the given presentation of \tilde{F} (in fact, it does not even depend on the imbedding $X \rightarrow U$, but we shall not need it here). For, suppose we have a second presentation

$$\Gamma(U, \mathcal{O}_U)^{q'} \xrightarrow{\alpha'} \Gamma(U, \mathcal{O}_U)^{p'} \xrightarrow{\beta'} \tilde{F} \rightarrow 0.$$

As $\Gamma(U, \mathcal{O}_U)^p$ is free over $\Gamma(U, \mathcal{O}_U)$, we can find a $\Gamma(U, \mathcal{O}_U)$ -linear map $\Gamma(U, \mathcal{O}_U)^p \xrightarrow{\gamma} \Gamma(U, \mathcal{O}_U)^{p'}$ such that $\beta = \beta' \circ \gamma$; this induces a continuous map

$$\Gamma(U, \mathcal{O}_U)^p / \text{Im } \Gamma(U, \alpha) \rightarrow \Gamma(U, \mathcal{O}_U)^{p'} / \text{Im } \Gamma(U, \alpha')$$

which is bijective, hence bicontinuous according to the closed graph theorem.

2. General case

If X is an analytic space and F an analytic coherent sheaf on X , we can find a) a locally finite covering of X by open subspaces X_i , b) for each i , a morphism $X_i \rightarrow U_i$, U_i open polycylinder in \mathbb{C}^{n_i} , which identifies X_i with a closed subspace of U_i c) for each i , a coherent sheaf \tilde{F}_i on U_i admitting a finite presentation, such that \tilde{F}_i is the extension of $F|_{X_i}$.

On $\Gamma(X_i, F|_{X_i})$ we have already defined a topology; further, consider the natural injection

$$\Gamma(X, F) \rightarrow \prod_i \Gamma(X_i, F|_{X_i})$$

We claim that its image is closed. For, (f_i) belongs to the image if and only if, for all $x \in X_i \cap X_j$ ($= X_i \times_X X_j$), we have $(f_i)_x = (f_j)_x$; and the fact that these relations define a closed subspace results easily from Krull's theorem.

This gives a topology of Fréchet space on $\Gamma(X, F)$. It does not depend on the chosen covering (if one has two coverings, one considers a common refinement, and one applies again Krull's theorem and the closed graph theorem; we leave the details to the reader). One proves in the same way that if X' is an open subspace of X , the restriction map $\Gamma(X, F) \rightarrow \Gamma(X', F|_{X'})$ is continuous. If X' is relatively compact in X , then the restriction map is compact (this can be seen by choosing a covering X'_j of X' of the same type, such that, for any j , there exist i with $X'_j \subset X_i$, X'_j relatively compact in X_i , and applying Ascoli's theorem).

4.3. Topology on $H^p(X, F)$

We consider a locally finite covering $\mathcal{U} = \{X_i\}_{i \in I}$ by open subspaces of the preceding type. If we have $i_0, \dots, i_p \in I$, we consider the natural morphisms

$$X_{i_0 \dots i_p} = X_{i_0} \times_X \dots \times_X X_{i_p} \rightarrow X_{i_0} \times \dots \times X_{i_p} \rightarrow U_{i_0} \times \dots \times U_{i_p}$$