

## 4.3. Topology on $H^p(X, F)$

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$$\Gamma(U, \mathcal{O}_U)^{q'} \xrightarrow{\alpha'} \Gamma(U, \mathcal{O}_U)^{p'} \xrightarrow{\beta'} \tilde{F} \rightarrow 0.$$

As  $\Gamma(U, \mathcal{O}_U)^p$  is free over  $\Gamma(U, \mathcal{O}_U)$ , we can find a  $\Gamma(U, \mathcal{O}_U)$ -linear map  $\Gamma(U, \mathcal{O}_U)^p \xrightarrow{\gamma} \Gamma(U, \mathcal{O}_U)^{p'}$  such that  $\beta = \beta' \circ \gamma$ ; this induces a continuous map

$$\Gamma(U, \mathcal{O}_U)^p / \text{Im } \Gamma(U, \alpha) \rightarrow \Gamma(U, \mathcal{O}_U)^{p'} / \text{Im } \Gamma(U, \alpha')$$

which is bijective, hence bicontinuous according to the closed graph theorem.

## 2. General case

If  $X$  is an analytic space and  $F$  an analytic coherent sheaf on  $X$ , we can find a) a locally finite covering of  $X$  by open subspaces  $X_i$ , b) for each  $i$ , a morphism  $X_i \rightarrow U_i$ ,  $U_i$  open polycylinder in  $\mathbb{C}^{n_i}$ , which identifies  $X_i$  with a closed subspace of  $U_i$  c) for each  $i$ , a coherent sheaf  $\tilde{F}_i$  on  $U_i$  admitting a finite presentation, such that  $\tilde{F}_i$  is the extension of  $F|_{X_i}$ .

On  $\Gamma(X_i, F|_{X_i})$  we have already defined a topology; further, consider the natural injection

$$\Gamma(X, F) \rightarrow \prod_i \Gamma(X_i, F|_{X_i})$$

We claim that its image is closed. For,  $(f_i)$  belongs to the image if and only if, for all  $x \in X_i \cap X_j (= X_i \times_X X_j)$ , we have  $(f_i)_x = (f_j)_x$ ; and the fact that this relations define a closed subspace results easily from Krull's theorem.

This gives a topology of Frechet space on  $\Gamma(X, F)$ . It does not depend on the chosen covering (if one has two coverings, one considers a common refinement, and one applies again Krull's theorem and the closed graph theorem; we leave the details to the reader). One proves in the same way that if  $X'$  is an open subspace of  $X$ , the restriction map  $\Gamma(X, F) \rightarrow \Gamma(X', F|_{X'})$  is continuous. If  $X'$  is relatively compact in  $X$ , then the restriction map is compact (this can be seen by choosing a covering  $X'_j$  of  $X'$  of the same type, such that, for any  $j$ , there exist  $i$  with  $X'_j \subset X_i$ ,  $X'_j$  relatively compact in  $X_i$ , and applying Ascoli's theorem).

### 4.3. Topology on $H^p(X, F)$

We consider a locally finite covering  $\mathcal{U} = \{X_i\}_{i \in I}$  by open subspaces of the preceding type. If we have  $i_0, \dots, i_p \in I$ , we consider the natural morphisms

$$X_{i_0 \dots i_p} = X_{i_0} \times_X \dots \times_X X_{i_p} \rightarrow X_{i_0} \times \dots \times X_{i_p} \rightarrow U_{i_0} \times \dots \times U_{i_p}$$

which makes  $X_{i_0}, \dots, i_p$  isomorphic with a closed subspace of  $U_{i_0} \times \dots \times U_{i_p}$  (the hypothesis that  $X$  is separated is essential here! See remark at the end of this paragraph), therefore,  $X_{i_0}, \dots, i_p$  satisfies theorems  $A$  and  $B$ ; more generally, if a finite number of open subspaces of  $X$  is Stein, their intersection is also Stein.

Introduce a total order on  $I$ . Given an analytic coherent sheaf on  $X$ , we can identify the alternating cochains of degree  $p$  of the covering  $\mathcal{U}$  with values in  $F$  with the space

$$C^p(\mathcal{U}, F) = \prod_{i_0 < i_1 < \dots < i_p} \Gamma(X_{i_0 \dots i_p}, F|_{X_{i_0 \dots i_p}}).$$

This is a Frechet space, and the differential  $d: C^p(\mathcal{U}, F) \rightarrow C^{p+1}(\mathcal{U}, F)$  is clearly continuous. Therefore the kernel  $Z^p(\mathcal{U}, F)$  is a closed subspace of  $C^p(\mathcal{U}, F)$ . We denote  $B^p(\mathcal{U}, F)$  the image of  $C^{p-1}(\mathcal{U}, F)$  under  $d$ , and we consider on  $H^p(\mathcal{U}, F) = Z^p(\mathcal{U}, F)/B^p(\mathcal{U}, F)$  the quotient topology; according to Leray's theorem, there is a natural isomorphism  $H^p(X, F) \simeq H^p(\mathcal{U}, F)$ .

This gives a topology on  $H^p(X, F)$  of a quotient of a Frechet space. In general, this topology is *not separated*.

We prove now that this topology is independent of the covering  $\mathcal{U}$ ; to do that, it is sufficient to consider a refinement  $\mathcal{U}' = \{X'_j\}_{j \in J}$  of  $\mathcal{U}$  of the same type, a map  $\varphi: J \rightarrow I$  such that  $X'_j \subset X_{\varphi(j)}$  for any  $j$  to consider the map defined by  $\varphi: C^*(\mathcal{U}, F) = \bigoplus_p C^p(\mathcal{U}, F) \xrightarrow{\varphi} C^*(\mathcal{U}', F)$  and to prove that the induced map  $\bar{\rho}: H^p(\mathcal{U}, F) \rightarrow H^p(\mathcal{U}', F)$  is an isomorphism.

First,  $\bar{\rho}$  is obviously continuous and bijective; so, according to the closed graph theorem, all that we have to prove is that  $\bar{\rho}$  maps the adherence of 0 onto the adherence of zero; to do that, we consider  $\bar{a}' \in H^p(\mathcal{U}, F)$ , which is adherent to zero; this means that  $\bar{a}'$  is the class modulo  $B^p(\mathcal{U}', F)$  of some  $a' \in Z^p(\mathcal{U}', F)$  which is adherent to  $B^p(\mathcal{U}', F)$ ; therefore, we have

$$a' = \lim_{n \rightarrow \infty} db'_n, \quad b'_n \in C^{p-1}(\mathcal{U}', F).$$

Now, the map

$$Z^p(\mathcal{U}, F) \oplus C^{p-1}(\mathcal{U}', F) \xrightarrow{(\rho, d)} Z^p(\mathcal{U}', F)$$

is surjective hence, according to the closed graph theorem, we can find converging sequences  $a_n \in Z^p(\mathcal{U}, F)$  and  $b''_n \in C^{p-1}(\mathcal{U}', F)$  such that  $db'_n = \rho(a_n) + db''_n$ ; but,  $\bar{\rho}$  being an isomorphism, we have  $\bar{a}_n = d\alpha_n$ ,  $\alpha_n \in C^{n-1}(\mathcal{U}, F)$ ; if we put  $b = \lim_{n \rightarrow \infty} b_n$ ,  $a = \lim_{n \rightarrow \infty} a_n$ , we find that  $a \in B^p(\mathcal{U}, F)$  and that the class  $a$  of  $a$  in  $H^p(\mathcal{U}, F)$  verifies  $\bar{\rho}(\bar{a}) = \bar{a}'$ ; this proves the result.

*Remark.* If  $X$  is not separated, an intersection of two open Stein subspaces of  $X$  need not be Stein; take f.i. for  $X$  two copies of  $\mathbb{C}^2$ , identified everywhere except at  $O$ ; there is an obvious covering of  $X$  by two open subspaces, identicals with  $\mathbb{C}^2$ ; but their intersection is  $\mathbb{C}^2 - \{O\}$ , and therefore is not Stein!

#### 4.4. The finiteness theorem

*Theorem 4.4.1.* (Cartan — Serre). Let  $X$  be a compact analytic space, and  $F$  be a coherent analytic sheaf on  $X$ . Then, for every  $p \geq 0$   $H^p(X, F)$  is separated and finite dimensional.

We shall give two proofs of this theorem ; both are interesting for further applications.

*1st proof.* Let  $\{X_i\}$  and  $\{X'_i\}$  be two finite coverings of  $X$  of the type considered in the previous articles, such that, for every  $i$ ,  $X'_i$  is relatively compact in  $X_i$ . Then, if we denote by  $\mathcal{U}$  (resp.  $\mathcal{U}'$ ) the covering  $\{X_i\}$  (resp.  $\{X'_i\}$ ), the natural restriction map  $C^p(\mathcal{U}, F) \rightarrow C^p(\mathcal{U}', F)$  is compact.

Consider now the map

$$(\rho, d) : Z^p(\mathcal{U}, F) \oplus C^{p-1}(\mathcal{U}', F) \rightarrow Z^p(\mathcal{U}', F)$$

this map is surjective, and we have  $(, d\rho) = (\rho, 0) + (0, d)$ ,  $(\rho, 0)$  being compact ; then the following lemma proves that  $\text{Im}(0, d)$  is closed and finite codimensional, q.e.d.

*Lemma 4.4.2.* Let  $E$  and  $F$  two Frechet spaces,  $u_1$  and  $u_2$  two linear continuous maps  $E \rightarrow F$  such that  $u_1 + u_2$  is surjective, and  $u_1$  compact. Then  $\text{Im}(u_2)$  is closed and finite codimensional. For the proof, see e.g. [5].

*2nd proof.* Consider  $\mathcal{U}$  and  $\mathcal{U}'$  as above, and consider the map  $(\rho, d) : C^{p-1}(\mathcal{U}, F)/Z^{p-1}(\mathcal{U}, F) \rightarrow [C^{p-1}(\mathcal{U}', F)/Z^{p-1}(\mathcal{U}', F)] \oplus Z^p(\mathcal{U}, F)$ .  $(\rho, d)$  is clearly injective. I claim that its image is closed: In fact, since  $\bar{\rho} : H^p(\mathcal{U}, F) \rightarrow H^p(\mathcal{U}', F)$  is injective, this image consists of the pairs  $(\bar{a}', b)$ ,  $a' \in C^{p-1}(\mathcal{U}', F)$ ,  $b \in Z^p(\mathcal{U}, F)$  such that  $da' = \rho b$ , which proves the assertion.

Now we have  $(\rho, d) = (\rho, 0) + (0, d)$  and  $(\rho, 0)$  is compact. By a well-known lemma, it results that  $\text{Im}(0, d)$  is closed, which means that  $H^p(\mathcal{U}, F)$  is separated.

Finally, since  $\bar{\rho}$  is compact, and is an isomorphism, it follows that the identity map of  $H^p(\mathcal{U}, F)$  into itself is compact ; therefore this space is finite dimensional ; this proves the theorem.