

## 4. Line bundle associated to a divisor

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easily seen that the sequence

$$0 \rightarrow H \rightarrow \mathcal{O}(X) \rightarrow \mathbf{C}_a \oplus \mathbf{C}_b \rightarrow 0$$

is exact. From this we conclude as above that there exists an integer  $s(a, b)$  such that the sequence

$$\Gamma(X, \underline{E}^{s(a,b)}) \rightarrow E_a^{s(a,b)} \oplus E_b^{s(a,b)} \rightarrow 0$$

is exact. Therefore there exists a neighbourhood  $W$  of  $(a, b)$  in  $X \times X$  such that if  $(a', b') \in W$ , then the sections of  $\Gamma(X, \underline{E}^{s(a,b)})$  separate  $a'$  and  $b'$ ; that is, if  $\sigma_0, \dots, \sigma_k$  is a basis of  $\Gamma(X, \underline{E}^{s(a,b)})$ , then  $(\sigma_0(a'), \dots, \sigma_k(a'))$  and  $(\sigma_0(b'), \dots, \sigma_k(b'))$  are different points in  $\mathbf{P}^k$ . Let  $l$  be a positive integer, let  $(a', b') \in W$ , and let  $\sigma$  be a section of  $\Gamma(X, \underline{E}^{s(a,b)})$  such that  $\sigma(a') \neq 0$  and  $\sigma(b') \neq 0$ . Then  $\sigma^{l-1} \otimes \sigma_0, \dots, \sigma^{l-1} \otimes \sigma_k$  are sections of  $\Gamma(X, \underline{E}^{ls(a,b)})$  such that  $((\sigma^{l-1} \otimes \sigma_0)(a'), \dots, (\sigma^{l-1} \otimes \sigma_k)(a'))$  and  $((\sigma^{l-1} \otimes \sigma_0)(b'), \dots, (\sigma^{l-1} \otimes \sigma_k)(b'))$  are different points in  $\mathbf{P}^k$ .

This means that for every positive integer  $l$  the sections of  $\Gamma(X, \underline{E}^{ls(a,b)})$  separate all point pairs in  $W$ . Thus, covering  $X \times X - U$  by finitely many such neighbourhoods and taking  $s''$  to be the product of the corresponding  $s(a, b)$ , we find that the sections of  $\Gamma(X, \underline{E}^{s''})$  separate all point pairs in  $X \times X - U$ .

Let  $\alpha = s's''$  and let  $\sigma_0, \dots, \sigma_d$  be a basis of  $\Gamma(X, \underline{E}^\alpha)$ . We claim that the mapping  $f$  from  $X$  into  $\mathbf{P}^d$  defined by  $f(x) = (\sigma_0(x), \dots, \sigma_d(x))$  is a biholomorphic imbedding of  $X$  into  $\mathbf{P}^d$ . That this mapping is regular follows from the fact that  $\alpha$  is a multiple of  $s'$ . What remains to be proved is that the mapping is injective.

Suppose  $a, b \in X$ ,  $a \neq b$ . If  $(a, b) \in U$ , then  $a, b \in U_i$  for some  $i$ , and since  $\alpha$  is a multiple of  $s'$ , we have  $f(a) \neq f(b)$ . If  $(a, b) \in X \times X - U$ , then  $f(a) \neq f(b)$  since  $\alpha$  is a multiple of  $s''$ . This proves the theorem.

#### 4. LINE BUNDLE ASSOCIATED TO A DIVISOR

Let  $X$  be a complex manifold and  $D$  an analytic subset of  $X$  of pure codimension 1 at every point. Such a set  $D$  is called a *divisor* of  $X$ . We shall construct a line bundle  $F$  on  $X$ , associated to  $D$ .

To do this, we observe that every point of  $X$  has a neighbourhood  $U$  in which there is a holomorphic function  $s$  such that  $U \cap D = \{x \in U; s(x) = 0\}$ , and  $s$  generates, at every point of  $U$ , the ideal of germs of holomorphic functions vanishing on  $D$ . Thus we get a covering of  $X$  by open sets  $U_j$  and

corresponding holomorphic functions  $s_j$ . The functions  $g_{ij} = s_i/s_j$  are then holomorphic and  $\neq 0$  on  $U_i \cap U_j$  and  $g_{ij} g_{jk} = g_{ik}$  on  $U_i \cap U_j \cap U_k$ . The functions  $g_{ij}$  therefore define a line bundle  $F$  on  $X$  with transition functions  $g_{ij}$  (see sect. 1). This bundle  $F$  is determined by  $D$  uniquely up to isomorphism.

If  $f \in \Gamma(X, F)$ , then the isomorphism  $F|U_j \simeq U_j \times \mathbf{C}$  gives a holomorphic function  $f_j$  on  $U_j$  corresponding to  $f$ . The functions  $f_j$  are related by  $f_i = g_{ij} f_j$  on  $U_i \cap U_j$ . Conversely, if  $f_j$  are holomorphic functions on  $U_j$ , satisfying this condition, then there is a section  $f$  of  $F$  on  $X$ , which corresponds to  $f_j$  on  $U_j$ . In particular, the  $s_j$  define a section  $s_D$  of  $F$  on  $X$ , and we have  $D = \{x \in X; s_D(x) = 0\}$ .

*Example.* Let  $X = \mathbf{P}^n$ , and let  $H$  be the hyperplane defined in the homogeneous coordinates  $z_0, \dots, z_n$  by  $z_0 = 0$ . Then the process above associates to  $H$  a line bundle  $F$  on  $\mathbf{P}^n$ . As defining functions we can use  $s_j(z_0, \dots, z_n) = z_0/z_j$  on the set  $U_j$  where  $z_j \neq 0$ , ( $j=0, \dots, n$ ). We shall prove that  $F$  is positive.

Each homogeneous coordinate  $z_k$  defines a section  $s^{(k)}$  of  $F$ , which on each  $U_j$  corresponds to the holomorphic function  $z_k/z_j$ , for the transition functions are  $g_{ij} = s_i/s_j = z_j/z_i$  and we have  $z_k/z_i = (z_k/z_j) g_{ij}$ . Now any section of  $F$  can be regarded as a holomorphic function on  $E = F^*$ , which is linear on the fibres of  $E$ . In particular,  $s^{(0)}, \dots, s^{(n)}$  give a holomorphic mapping  $\varphi: E \rightarrow \mathbf{C}^{n+1}$ . It is clear that the zero section in  $E$  is equal to  $\varphi^{-1}(0)$ . It is seen by direct verification that  $\varphi$  maps  $E$  onto  $\mathbf{C}^{n+1}$  and  $E - \varphi^{-1}(0)$  biholomorphically onto  $\mathbf{C}^{n+1} - \{0\}$ . Hence  $E$  is negative and  $F$  is positive (see sect. 1).

If  $V$  is a submanifold of  $\mathbf{P}^n$ , then the restriction of  $F$  to  $V$  is a positive line bundle associated to the hyperplane section  $D = V \cap H$ . In fact, the dual of the restriction is the restriction  $E|V$  of  $E$  to  $V$ , and we can use the restriction of  $\varphi$  to  $E|V$  as "blowing down mapping".

Let again  $X$  be a complex manifold,  $D$  a divisor of  $X$ , and  $F$  the line bundle on  $X$ , associated to  $D$ . What are the sections of  $F^k$ ?

If  $U \in \Gamma(X, F^k)$ , then  $s$  is represented in local coordinates on  $U_j$  by a holomorphic function  $f_j$ . The  $f_j$  are connected by  $f_i = g_{ij}^k f_j$  on  $U_i \cap U_j$ , because the functions  $g_{ij}^k$  are transition functions for  $F^k$ . Now  $s_i^k = g_{ij}^k s_j^k$  on  $U_i \cap U_j$ , the  $s_i$  being local equations for the set  $D$  as above, and thus  $f_i/s_i^k = f_j/s_j^k$  on  $U_i \cap U_j$ . Hence there exists a meromorphic function  $f$  on  $X$  such that  $f_j = s_j^k f$  on  $U_j$ .

This means that  $f$  is meromorphic with poles only on  $D$  and of order  $\leq k$ . Conversely, if  $f$  is such a meromorphic function, then  $f_j = s_j^k f$  are holomorphic on  $U_j$  and satisfy  $f_i = g_{ij}^k f_j$  on  $U_i \cap U_j$ . Therefore they give a section  $s$  of  $F^k$ . This correspondence is obtained simply by associating to the section  $u$  of  $F^k$ , the meromorphic function  $u \otimes s_D^{-k}$ .

Let us consider again the space  $\mathbf{P}^n$  and the bundle  $F$  associated to a hyperplane section. Let  $(z_0, \dots, z_n)$  denote homogeneous coordinates for  $\mathbf{P}^n$ . If  $u \in \Gamma(\mathbf{P}^n, F^k)$ ,  $u$  defines, for  $z \in \mathbf{P}^n$ , an element of  $F_z = (E_z^*)^k$ ,  $E$  being the dual bundle to  $F$ , hence a map of  $E_z$  into  $\mathbf{C}$  which is homogeneous of degree  $k$ . Thus,  $u$  defines a map  $\hat{u}$  of  $E \rightarrow \mathbf{C}$ , homogeneous of degree  $k$  on each fibre. If  $\varphi$  denotes the map of  $E$  into  $\mathbf{C}^{n+1}$  defined above,  $\hat{u}: E \rightarrow \mathbf{C}$  is holomorphic, and vanishes on  $\varphi^{-1}(0)$ , and so defines a holomorphic function  $v$  on  $\mathbf{C}^{n+1}$  which is homogeneous of degree  $k$  ( $v$  is holomorphic also at 0 since a continuous function holomorphic outside a point in  $\mathbf{C}^{n+1}$ ,  $n \geq 1$ , is holomorphic also at this point). The Taylor expansion of  $v$  about 0 shows that  $v$  is a homogeneous polynomial of degree  $k$ . Thus, any  $u \in \Gamma(\mathbf{P}^n, F^k)$  can be identified with a homogeneous polynomial of degree  $k$  in the homogeneous coordinates  $(z_0, \dots, z_n)$  [i.e. the sections  $s^{(0)}, \dots, s^{(n)}$  of  $F$  defined above].

As an application of the vanishing theorem of Kodaira, we now prove the following result due to Chow (cf. [3], p. 170).

*Theorem 4.1.* Let  $A$  be a subvariety of  $\mathbf{P}^n$ . Then there exist homogeneous polynomials  $f_1, \dots, f_k$  such that  $A = \{a \in \mathbf{P}^n; f_1(a) = \dots = f_k(a) = 0\}$ .

*Proof.* We first prove that if  $b \notin A$ , then there exists a homogeneous polynomial  $f$  vanishing on  $A$  with  $f(b) \neq 0$ . Let  $S$  be the sheaf of germs of holomorphic functions vanishing on  $A$  and let  $I$  be the sheaf of germs of holomorphic functions vanishing at  $b$ . Let  $F$  be the line bundle associated to a hyperplane section of  $A$ . Then  $F$  is positive. We get an exact sequence

$$0 \rightarrow I \otimes S \otimes F^m \rightarrow S \otimes F^m \rightarrow S_b \otimes F_b^m \rightarrow 0.$$

By the vanishing theorem of Kodaira, part of the corresponding cohomology sequence will be

$$H^0(\mathbf{P}^n, S \otimes F^m) \rightarrow H^0(\mathbf{P}^n, S_b \otimes F_b^m) \rightarrow 0,$$

if  $m$  is sufficiently large. Thus there exists  $f \in H^0(\mathbf{P}^n, S \otimes F^m)$  which is not zero at  $b$ . Since  $S \subset \mathcal{O}$ , we may look upon  $H^0(S \otimes F^m)$  as a subspace of  $H^0(F^m)$ . It is then the subspace of those sections of  $H^0(F^m)$  which vanish

on  $A$ . Since  $f \in H^0(\mathbf{P}^n, F^m)$ , this gives the desired homogeneous polynomial.

To prove the theorem, it now suffices to consider all homogeneous polynomials which vanish on  $A$  without being identically zero and apply the Hilbert basis theorem.

## 5. MEROMORPHIC FORMS

Let  $X$  be a complex manifold. A holomorphic differential form is a form which in local coordinates can be written as a finite sum

$$\omega = \sum a_{i_1 \dots i_k} dz_{i_1} \wedge \dots \wedge dz_{i_k} \quad (5.1)$$

with holomorphic coefficients  $a_{i_1 \dots i_k}$ .

A form is called meromorphic if it has locally the form (5.1) with coefficients that are meromorphic functions. Every meromorphic function can be written locally as  $f\omega$  where  $f$  is a meromorphic function and  $\omega$  a holomorphic form. The exterior differentiation  $d$ , satisfying  $d^2 = 0$ , extends naturally to meromorphic forms.

Let  $D$  be a divisor of  $X$  and let  $\Omega^p(k, D) = \Omega^p(X, k, D)$  be the sheaf of germs of meromorphic  $p$ -forms on  $X$  with poles only on  $D$  and of order  $\leq k$ , and let  $\Omega^p = \Omega^p(X)$  be the sheaf of germs of holomorphic  $p$ -forms on  $X$ .

*Lemma 5.1.* There is a natural isomorphism

$$\Omega^p(k, D) \simeq \Omega^p \otimes \underline{F^k}.$$

*Proof.* A germ in  $\Omega^p(k, D)$  at  $a \in X$  is represented by a form  $f\omega$ , where  $f$  is a meromorphic function in a neighbourhood  $U$  of  $a$ , with poles only on  $D$  and of order  $\leq k$ , and  $\omega$  is a holomorphic form on  $U$ . Now to  $f$  corresponds biuniquely a section  $s \in \Gamma(U, F^k)$  (see Sect. 4), which gives a germ  $s_a \in \underline{F^k}_a$ . Also  $\omega$  defines a germ  $\omega_a \in \Omega^p_a$ .

The desired mapping  $\Omega^p(k, D) \rightarrow \Omega^p \otimes \underline{F^k}$  is now uniquely defined by

$$f\omega \rightarrow \omega_a \otimes s_a.$$

To see that it is an isomorphism, it is sufficient to observe that the inverse mapping of  $\Omega^p \otimes \underline{F^k}$  into  $\Omega^p(k, D)$  is induced by the bilinear mapping  $\Omega^p \oplus \underline{F^k} \rightarrow \Omega^p(k, D)$ , which is given by

$$(\omega_a, s_a) \rightarrow (f\omega)_a, \quad (a \in X).$$