7. Lefschetz' theorem on hyperplane sections

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Now ω takes the form

$$\omega - d\beta' = dz_1 \wedge \alpha' . \tag{6.23}$$

We distinguish the two cases $k > 1$ and $k = 1$. In the first case we get from (6.23)

$$dz_1 \wedge \delta\alpha' = 0 ,$$

which implies that $\delta\alpha' = 0$. Since α' is a form of type $q-1 \geqslant 1$, we can apply once again Lemma 6.8 and get $\alpha' = \delta\alpha''$. Thus $dz_1 \wedge \alpha' = d(dz_1 \wedge \alpha'')$, and we get $\omega = d(\beta' + dz_1 \wedge \alpha'')$. This proves that the cohomology under consideration is trivial for $k > 1$.

Finally, in the case $k = 1$, α' is a meromorphic function, independent of $z_2, \ldots, z_n$. Thus by (6.23), $\omega = d\gamma$ for some γ if and only if in the Laurent expansion of α' the coefficient of z_1^{-1} is zero. Thus the cohomology in dimension 1 is generated by $z_1^{-1}\, dz_1$, which completes the proof of Theorem 6.4.

7. Lefschetz' theorem on hyperplane sections

The Lefschetz theorem in the slightly more general setting proved by Andreotti and Frankel [1], is the following:

Theorem 7.1. Let V be a submanifold of $\mathbf{P}^n$ of complex dimension d and let D be a hyperplane section of V (not necessarily non-singular). Then there are natural isomorphisms

$$H^q(V, \mathbf{Z}) \simeq H^q(D, \mathbf{Z}) , \qquad (\forall q < d-1) ,$$

and a natural injection

$$H^{d-1}(V, \mathbf{Z}) \to H^{d-1}(D, \mathbf{Z}) .$$

Proof. $X = V - D$ is a Stein manifold, since it is imbedded as a closed submanifold of $\mathbf{C}^n$. Now one knows that

$$H^q(V, D, \mathbf{Z}) \simeq H^q_c(X, \mathbf{Z}) , \tag{7.1}$$

where the c indicates cohomology with compact support. On the other hand, since X is a topological manifold of dimension $2\,d$, Poincaré duality gives

$$H^q_c(X, \mathbf{Z}) \simeq H_{2d-q}(X, \mathbf{Z}) . \tag{7.2}$$

Now we shall use the following theorem:

Theorem 7.2. Let X be a Stein manifold of dimension d. Then

$$H_r(X, \mathbf{Z}) = 0, \quad (\forall r > d). \tag{7.3}$$

Suppose this theorem is proved. Then (7.1) – (7.3) gives

$$H^q(V, D, \mathbf{Z}) = 0. \quad (\forall q < d). \tag{7.4}$$

Now we have the exact sequence

$$\ldots \to H^q(V, D, \mathbf{Z}) \to H^q(V, \mathbf{Z}) \to H^q(D, \mathbf{Z}) \to H^{q+1}(V, D, \mathbf{Z}) \to \ldots,$$

and using (7.4) we conclude that the mapping

$$H^q(V, \mathbf{Z}) \to H^q(D, \mathbf{Z})$$

is an isomorphism onto when $q < d-1$ and an injection when $q = d-1$.

This proves Lefschetz' theorem.

The proof of Theorem 7.2 is based on *Morse theory.* Let X be a C^∞-manifold with countable base. If f is a real-valued C^∞-function on X, then a point $a \in X$ is called *critical* for f if $df(a) = 0$. A critical point a is *non-degenerate*, if in local coordinates $f(x) - f(a) = \Sigma a_{ij}(x_i - a_i)(x_j - a_j) + o(|x-a|^2)$, where the symmetric matrix (a_{ij}) is non-singular. It is non-degenerate of index r if (a_{ij}) has r eigenvalues < 0. The non-degenerate critical points for f are necessarily isolated. We now quote some facts from Morse theory; for proofs, see [6].

Lemma 7.3. Suppose that $f \in C^\infty(X)$, $f \geqslant 0$, $\alpha < \beta$, and that $X_\beta = \{x \in X; f(x) \leqslant \beta\}$ is compact.

(a) If f has no critical points in $\{x \in X: \alpha \leqslant f(x) \leqslant \beta\}$, then X_α is a deformation retract of X_β, and hence

$$H_r(X_\beta, X_\alpha, \mathbf{Z}) = 0, \quad (\forall r \geqq 0).$$

(b) If all critical points of f in $\{x \in X;\ \alpha \leqslant f(x) \leqslant \beta\}$ are non-degenerate of index $\leqslant d$, then

$$H_r(X_\beta, X_\alpha, \mathbf{Z}) = 0, \quad (\forall r > d).$$

In particular, if all critical points of f in X_β are non-degenerate of index $\leqslant d$, then

$$H_r(X_\beta, \mathbf{Z}) = 0, \quad (\forall r > d).$$

In the proof of Theorem 7.2 we shall also use the following lemma of Morse:

Lemma 7.4. Let X be a C^∞-manifold with countable base. Then every real function $g \in C^\infty(X)$ can be approximated in the topology of $C^\infty(X)$ by real functions $f \in C^\infty(X)$, whose critical points are all non-degenerate.

The topology of $C^\infty(X)$ is the topology of uniform convergence of all derivatives on compact sets. Therefore the lemma explicitly means the following:

Let $\varepsilon > 0$, an integer $r \geqslant 0$ and a compact set $K \subset X$ be given, and let $K = K_1 \cup \ldots \cup K_k$, where each K_j is compact and contained in an open set U_j, where we have a coordinate system. Then there is a function f of the prescribed type such that

$$\sup_{j} \sup_{|\alpha| \leq r} \sup_{x \in K_j} |D^\alpha f(x) - D^\alpha g(x)| < \varepsilon .$$

(Here D^α means a derivative of order $|\alpha|$ in the coordinates on U_j.)

To prove Lemma 7.4 we shall use a Lemma of Sard (see [8, Ch. I,§3, Th. 4]):

Lemma 7.5. Let Ω be an open subset of $\boldsymbol{R}^n$ and $f: \Omega \to \boldsymbol{R}^n$ a C^1-mapping. Let A be the critical set of f, i.e. the set of $a \in \Omega$ where $\det(\partial f_i(a)/\partial x_j) = 0$. Then $f(A)$ has Lebesgue measure 0 in $\boldsymbol{R}^n$. In particular, $f(A)$ is nowhere dense in $\mathbf{R}^n$.

Proof of Lemma 7.4. Suppose first that X is an open subset Ω of $\mathbf{R}^n$. If $g \in C^\infty(\Omega)$ is realvalued, consider the mapping

$$\varphi : \Omega \ni x \to (\partial g/\partial x_1, \ldots, \partial g/\partial x_n) \in \mathbf{R}^n .$$

The critical set A of φ is the set in Ω where

$$\det(\partial^2 g/\partial x_i \, \partial x_j) = 0 .$$

The lemma of Sard, applied to φ, shows that there are arbitrarily small $\varepsilon_1, \ldots, \varepsilon_n \in \mathbf{R}$ such that $(\varepsilon_1, \ldots, \varepsilon_n) \notin \varphi(A)$. Put

$$f(x) = g(x) - \varepsilon_1 x_1 - \ldots - \varepsilon_n x_n .$$

A point $x \in \Omega$ is a critical point of f if and only if $\partial g/\partial x_j = \varepsilon_j$, $(j=1, \ldots, n)$.

At such points $\varphi(x) = (\varepsilon_1, \ldots, \varepsilon_n) \in \varphi(A)$ and hence $\det(\partial^2 g/\partial x_i \, \partial x_j) \neq 0$. Hence all critical points of f are non-degenerate.

Since $\varepsilon_1, \ldots, \varepsilon_n$ can be chosen arbitrarily small, the lemma is proved in the case $X = \Omega$.

The general case now follows by a category argument. From the special case we conclude that we can cover X by denumerably many relatively compact open subsets U_j of X, such that $\mathscr{U}_j$ is dense in the space of real C^∞-functions, where $\mathscr{U}_j$ denotes the set of real C^∞-functions, whose critical points in $\overline{U}_j$ are all non-degenerate. It is also easy to see that every $\mathscr{U}_j$ is open in the space of real C^∞-functions. Since this space is a real Fréchet space, we can therefore use Baire's theorem to conclude that the set of all real C^∞-functions, whose critical points in X are all non-degenerate, i.e. $\cap\, \mathscr{U}_j$, is dense. This proves the lemma of Morse.

Proof of Theorem 7.2. Let X be a Stein manifold of dimension d, and let K be a compact subset of X such that

$$K = \{x \in X;\ |f(x)| \leq \|f\|_K, \quad \forall f \text{ holomorphic on } X\}.$$

(Since X is a Stein manifold, every compact subset of X is contained in some K of this kind.) Choose an open set U such that $K \subset U \subset\subset X$. For every $a \in \partial U$ we can find a holomorphic function f on X such that $|f(x)| \geqslant 1$ in a neighbourhood of a and $\|f\|_K < 1$. Since ∂U is compact, we can therefore choose holomorphic functions $f_1, \ldots, f_k$ on X such that

$$\max |f_j(a)| \geq 1, \qquad (\forall a \in \partial U),$$

and

$$\|f_j\|_K < 1, \qquad (\forall j).$$

By replacing each f_j by a sufficiently high power, we can also arrange that the function

$$p(x) = \Sigma\, |f_j(x)|^2$$

satisfies $p(x) < 1$ on K and $p(x) \geqslant 1$ on ∂U. We can also assume that the rank of $(f_1, \ldots, f_k)$ is maximal at all points of U.

Now $p \in C^\infty(X)$, $p \geqslant 0$, and $U_\beta = \{x \in U; p(x) \leqslant \beta\}$ is compact and contains K if $\beta < 1$ is chosen so that $p(x) < \beta$ in K. By calculating the Levi form and using the maximality of the rank of $(f_1, \ldots, f_k)$, we see that p is strongly plurisubharmonic.

Because of Morse's lemma we can also assume that all critical points of p in U_β are non-degenerate. We shall prove that they are all of index $\leqslant d$.

We expand p at a critical point $a \in U_\beta$ in a local coordinate system:

$$\begin{aligned} p(x) &= p(a) + 2\mathrm{Re} \sum \frac{\partial^2 p(a)}{\partial z_i\, \partial z_j} (z_i - a_i)(z_j - a_j) \\ &\quad + \sum \frac{\partial^2 p(a)}{\partial z_i\, \partial \bar{z}_j} (z_i - a_i)(\bar{z}_j - \bar{a}_j) + \ldots \\ &= p(a) + \mathrm{Re}\, Q(z-a) + L(z-a) + \ldots . \end{aligned}$$

Here $L(z-a)$ is the Levi form of p at the point a. Now, since p is strongly plurisubharmonic, we can choose the coordinates so that $L(z-a) = |z-a|^2$. Then we see that if ζ is an eigenvector corresponding to an eigenvalue < 0 of the symmetric matrix of the real quadratic form Re $Q(z) + L(z)$, then $i\zeta$ is an eigenvector corresponding to an eigenvalue > 0. Hence the number of negative eigenvalues is $\leqslant d$, since the real dimension of X is $2d$. Thus the index of the critical point a is $\leqslant d$.

Now using Lemma 7.3 (b), we see that

$$H_r(U_\beta, \mathbf{Z}) = 0, \quad (\forall r > d).$$

From this it follows that

$$H_r(X, \mathbf{Z}) = 0, \quad (\forall r > d),$$

because the singular cycles defining the homology groups $H_r(X, \mathbf{Z})$ have compact supports, and any compact subset of X is contained in some compact set K with a corresponding $U_\beta \supset K$.

A refinement of the above argument leads to the stronger (homotopy) statement:

Any Stein manifold of (complex) dimension d has the same homotopy type as a CW complex of (real) dimension $\leqslant d$. (See [6]).

Moreover, the Lefschetz theorem has an analogue in homology and in homotopy [6]. The latter, for example, asserts that, if V, D are as in Th. 7.1, then the relative homotopy groups $\pi_q(V, D) = 0$ for $q < d$.

Th. 7.2 has been generalised in various directions. It has a relative analogue (relative to a Runge domain). Further, Th. 7.2 remains true if X is any Stein space (with singularities) of complex dimension d, but the corresponding cohomology statement is proved only for some other coefficient groups [5, 7]. Note that in view of the use of Poincaré duality, this does not lead to a Lefschetz theorem for algebraic varieties with singularities.

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