General Theory

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tion because $\overline{U^*_{\iota_0 \dots \iota_\lambda}} \subset \overline{U^*_{\iota_0}} \cap \dots \cap \overline{U^*_{\iota_\lambda}} \subset U_{\iota_0 \dots \iota_\lambda}$. In the same way we can prove that condition 3) is satisfied if ρ_2 is sufficiently small and the theorem is clear.

General Theory

Let G be an analytic manifold. We put $\hat{G} = G \times E^n(\rho_1)$ where ρ_1 is an n-tuple of positive numbers. Let $\pi \colon \hat{G} \to E^n(\rho_1)$ and $\mathfrak{P} \colon \hat{G} \to G$ be the projection maps. $\hat{G}^* \subset \hat{G}$ denotes an open subset and $G^* = \hat{G}^* \cap G \times \{0\}$.

The set G^* can be identified with an open subset of G. We denote by $\alpha \colon G^* \times E^n(\rho_1) \to \hat{G}^*$ a biholomorphic fiber preserving map, i.e. $\pi \circ \alpha = \pi^*$ where $\pi^* \colon G^* \times E^n(\rho_1) \to E^n(\rho_1)$ is the natural projection. Let $\rho \leqslant \rho_2 = \gamma\rho_1 < \rho_1$ where $0 < \gamma < 1$ is a fixed number. We put $\hat{G}(\rho) = G \times E^n(\rho)$. If f is a holomorphic function on $\hat{G}(\rho)$ we write $f = \sum a_\nu (t/\rho)^\nu$ with $a_\nu \in I(G)$. We define the norm $\|f\|_\rho$ of f by $\|f\|_\rho = \sup_\nu \{\sup |a_\nu(G)|\}$. If $f \in I(\hat{G}(\rho))$ we see that $f \circ \alpha$ is a well defined function on $G^* \times E^n(\rho)$ because α is fiber preserving. We define $\|f \circ \alpha\|_\rho$ using G^* instead of G as above. We have the proposition:

Proposition 1. There exists a constant K such that $\|f \circ \alpha\|_\rho \leqslant K \|f\|_\rho$ where $K = K(\rho_2)$ is independent of $\rho \leqslant \rho_2$.

Proof. We write $f = \sum_{|\nu|=0}^{\infty} a_\nu (t/\rho)^\nu$ with $a_\nu \in I(G)$. Now we get $f \circ \alpha = \sum (a_\nu \circ \mathfrak{P} \circ \alpha)(t/\rho)^\nu$ because α is fiber preserving. Since $\mathfrak{P}(\hat{G}^*) \subset G$ we get $|a_\nu \circ \mathfrak{P}(\hat{G}^*)| \leqslant |a_\nu(G)| \leqslant \|f\|_\rho$. Now $a_\nu \circ \mathfrak{P} \circ \alpha$ admits a Taylor series: $a_\nu \circ \mathfrak{P} \circ \alpha = \sum C_{\nu\lambda} (t/\rho)^\lambda$ with $C_{\nu\lambda} \in I(G^*)$. Since $|\sum C_{\nu\lambda}(t/\rho)^\lambda| \leqslant \|f\|_\rho$ in $G^* \times E^n(\rho_1)$ and $\rho \leqslant \rho_2 = \gamma\rho_1$ Cauchy's inequalities give us $|C_{\nu\lambda}(G^*)| \leqslant \|f\|_\rho \gamma^{|\lambda|}$. Let us put $b_\mu = \sum_{\nu+\lambda=\mu} C_{\nu\lambda}$. We get $|b_\mu(G^*)| \leqslant \|f\|_\rho \sum \gamma^{|\lambda|} = \|f\|_\rho (1-\gamma)^{-n} = K\|f\|_\rho$. Now we can write $f \circ \alpha = \sum_\nu a_\nu \circ \mathfrak{P} \circ \alpha (t/\rho)^\nu = \sum_{\lambda,\nu} C_{\nu\lambda} (t/\rho)^\lambda (t/\rho)^\nu = \sum_\mu b_\mu (t/\rho)^\mu$. By definition we have $\|f \circ \alpha\|_\rho = \sup_\mu |b_\mu(G^*)| \leqslant K\|f\|_\rho$.

Let us now consider $\mathbf{h} = (h_{\nu\mu})$ which is a $q \times q$ matrix with $h_{\nu\mu} \in I(\hat{G})$. The $h_{\nu\mu}$ are also assumed to be bounded on $\hat{G}$.

Proposition 2. Let $\mathbf{f} = (f_1 \dots f_q) \in qI(\hat{G}(\rho))$. Then $\| \mathbf{h}(\mathbf{f}) \|_\rho \leqslant K \| \mathbf{f} \|_\rho$. As before $\rho \leqslant \rho_2 = \gamma\rho_1 < \rho_1$ and K only depends on ρ_2.

Proof. We have $\mathbf{h}(\mathbf{f}) = (g_1 \dots g_q)$ with $g_\nu = \sum_\mu h_{\nu\mu} f_\mu$. Let us write $h_{\nu\mu} = \sum_\lambda a_{\nu\mu\lambda} (t/\rho)^\lambda$. By assumption $| h_{\nu\mu}(\hat{G}) | \leqslant M$ for some constant M and hence we have, by Cauchy's inequalities, $| a_{\nu\mu\lambda}(G) | \leqslant M\gamma^{|\lambda|}$. Let us also write $f_\mu = \sum_\lambda b_{\mu\lambda} (t/\rho)^\lambda$. By definition $\sup_{\mu,\lambda} | b_{\mu\lambda}(G) | = \| \mathbf{f} \|_\rho$. Now we get $g_\nu = \sum_\mu \sum_{\lambda_1, \lambda_2} a_{\nu\mu\lambda_1} b_{\mu\lambda_2} (t/\rho)^{\lambda_1 + \lambda_2} = \sum C_{\nu\lambda} (t/\rho)^\lambda$ where $C_{\nu\lambda} = \sum_\mu \sum_{\lambda_1 + \lambda_2 = \lambda} a_{\nu\mu\lambda_1} b_{\mu\lambda_2}$. We get easily $| C_{\nu\lambda}(G) | \leqslant qM \| \mathbf{f} \|_\rho (1-\gamma)^{-n} = K \| \mathbf{f} \|_\rho$. Hence $\| \mathbf{h}(\mathbf{f}) \|_\rho = \sup_\nu \| g_\nu \|_\rho = \sup_{\nu,\lambda} | C_{\nu\lambda}(G) | \leqslant K \| \mathbf{f} \|_\rho$.

We shall now apply these two propositions to our situation. Let $G^* \subset G \subset W_{\iota_0 \dots \iota_\lambda} \subset X_0$. Here G^* and G are open sets and $W_{\iota_0 \dots \iota_\lambda}$ comes from the measure atlas $\mathscr{W}$. As before $\rho \leqslant \rho_2 < \rho_* = \min \rho_\iota$. We are given ι and ι' from $\{ \iota_0, \dots, \iota_\lambda \}$ and the following inclusions are assumed: $(G^*)_{\iota'}(\rho_1) \subset (G)_\iota(\rho_1)$, $(G^*)_{\iota'}(\rho_1) \subset\subset \hat{W}_{\iota'}$, $(G)_\iota(\rho_1) \subset\subset \hat{W}_\iota$.

The following theorem is very important.

Theorem I. Let $S \in \Gamma((G)_\iota(\rho), \mathrm{F})$. Then $\| S | (G^*)_{\iota'}(\rho) \|_{\iota'} \leqslant K \| S \|_\iota$. K depends only on ρ_2.

Proof. We have the following diagram:

$$\begin{array}{ccc} (G)_\iota(\rho_1) & \xrightarrow{\Phi_\iota} & G \times E^n(\rho_1) \\ \text{injection} \uparrow & & \uparrow \alpha \\ (G^*)_{\iota'}(\rho_1) & \xrightarrow{\Phi_{\iota'}} & G^* \times E^n(\rho_1) \end{array}$$

α being a fiber preserving holomorphic map. We identify $S | (G^*)_{\iota'}(\rho)$ with an element of $qI(G^* \times E^n(\rho))$ using the trivialization of F in the chart $\mathscr{W}_{\iota'}$. Call this element S^*. Also S itself is considered as an element of $qI(G \times E^n(\rho))$ using the trivialization in the chart $\mathscr{W}_\iota$. Now we have $S^* = \mathbf{h}(S \circ \alpha)$ where $\mathbf{h}$ is a $q \times q$ matrix. The elements of $\mathbf{h}$ are holomorphic functions defined on $\Phi_{\iota'}(\hat{W}_{\iota\iota'}) \supset\supset G^* \times E^n(\rho_1)$. Hence the elements of $\mathbf{h}$ are bounded on $G^* \times E^n(\rho_1)$. It is now obvious how we can use 1) and 2) to finish the proof.

We shall need one more general result. Let G be an analytic manifold. G is assumed to be Stein and $R^* = \{ U_1, \dots, U_{\iota^*} \}$ a Stein covering of G.

The set $G^* \subset G$ is open and $R^{**} = \{ V_1, ..., V_{\iota^*} \}$ an open covering of G^* such that $V_\iota \subset\subset U_\iota$ for $\iota \in \{ 1, ..., \iota^* \}$. We have:

Cartan's Theorem. There exists a constant K such that if $\xi \in Z^l(R^*, q\mathcal{O})$ then $\xi \mid R^{**} = \delta\eta$ where $\eta \in C^{l-1}(R^{**}, q\mathcal{O})$ and $\| \eta \| \leqslant K \| \xi \|$ for $l \geqslant 1$.

This is a simple consequence of Theorem B and Banach's open mapping theorem.

Now we apply Cartan's theorem. We keep the notations as above. Let $\hat{G} = G \times E^n(\rho)$ and put $\hat{R}^* = \{ U_\iota \times E^n(\rho) \}$. Now $\hat{R}^*$ is a Stein covering of $\hat{G}$. Let $\hat{G}^* = G^* \times E^n(\rho)$ and $\hat{R}^{**} = \{ V_\iota \times E^n(\rho) \}$. Let $\hat{\xi} \in Z^l(\hat{R}^*, q\mathcal{O})$ and write $\hat{\xi} = \sum \xi_{(\nu)} (t/\rho)^\nu$ with $\xi_{(\nu)} \in Z^l(R^*, q\mathcal{O})$. We assume $\| \hat{\xi} \|_\rho = \sup\limits_\nu \| \xi_{(\nu)} \| < \infty$. Now Cartan's theorem gives $\xi_{(\nu)} \mid R^{**} = \delta\eta_\nu$ with $\eta_\nu \in C^{l-1}(R^{**}, q\mathcal{O})$ and $\| \eta_\nu \| \leqslant K \| \xi_{(\nu)} \| < \infty$. It follows that $\hat{\eta} = \sum \eta_\nu (t/\rho)^\nu$ is well defined in $C^{l-1}(\hat{R}^{**}, q\mathcal{O})$ and by definition we have $\| \hat{\eta} \|_\rho \leqslant K \| \hat{\xi} \|_\rho$.

Smoothing

We are given a sequence of admissible refinements of measure coverings in $X(\rho_1)$. Here $\rho_1 < \rho_0 = \min \rho_\iota$ as usual. Let l be a fixed integer $\geqslant 1$. We are given $\mathfrak{B}^* \ll \mathfrak{B}' = \mathfrak{B}_{3l} \ll \mathfrak{B}_{3l-1} \ll ... \ll \mathfrak{B}_1 \ll \mathfrak{B}_0 \ll \mathfrak{B} \ll \mathfrak{U}^* \ll \mathfrak{U} = \mathfrak{U}_{3l} \ll ... \ll \mathfrak{U}_0 \ll \mathfrak{U}'$. Here it is also required that $(\mathfrak{B}_{\nu+1}, \mathfrak{U}_{\nu+1}) \ll (\mathfrak{B}_\nu, \mathfrak{U}_\nu)$; $(\mathfrak{B}^*, \mathfrak{U}^*) \ll (\mathfrak{B}', \mathfrak{U})$ and $(\mathfrak{B}_0, \mathfrak{U}_0) \ll (\mathfrak{B}, \mathfrak{U}')$. These extra conditions mean: 1) $\hat{U}^{(\nu+1)}_{i_0 \dots i_\kappa} \cap \hat{V}^{(\nu+1)}_{\iota_0 \dots \iota_l} \subset (U^{(\nu)}_{i_0 \dots i_\kappa} \cap V^{(\nu)}_{\iota_0 \dots \iota_l})_i$ for each $i \in \{ i_0, ..., i_\kappa \}$ and 2) $(U^{(\nu+1)}_{i_0 \dots i_\kappa} \cap V^{(\nu+1)}_{\iota_0 \dots \iota_l})_j \subset (U^{(\nu)}_{i_0 \dots i_\kappa} \cap V^{(\nu)}_{\iota_0 \dots \iota_l})_i$ for all $i, j \in \{ i_0, \dots i_\kappa, \iota_0, \dots \iota_l \}$. Recall that all operations are done with respect to ρ_1. Let us put $\hat{R}^{(\nu)}_{i_0 \dots i_k \iota_0 \dots \iota_\kappa} = \hat{U}^{(\nu)}_{i_0 \dots i_k} \cap \hat{V}^{(\nu)}_{\iota_0 \dots \iota_\kappa}$. We consider elements $\xi_{i_0 \dots i_k \iota_0 \dots \iota_\kappa} \in \hat{\Gamma}(\hat{R}^{(\nu)}_{i_0 \dots i_k \iota_0 \dots \iota_\kappa}, \mathbf{F})$. Now we take a full collection $\hat{\xi} = \{ \hat{\xi}_{i_0 \dots i_k \iota_0 \dots \iota_\kappa} \}$ of such elements which is anticommutative in $\{ i_0, \dots i_k \}$ and $\{ \iota_0, ..., \iota_\kappa \}$. In this way we get a double complex $C_\nu^{k,\kappa}$. Here $\delta : C_\nu^{k,\kappa} \to C_\nu^{k+1,\kappa}$ and $\partial : C_\nu^{k,\kappa} \to C_\nu^{k,\kappa+1}$ are the usual coboundary operators.

Norm in $C_\nu^{k,\kappa}$: Let $\hat{\xi} \in C_\nu^{k,\kappa}$; we put