Approximation

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We use positive n-tuples $\rho, \ldots$ with $\rho \leqslant \rho_2 < \rho_3 < \rho_4 < \rho_1$ and $\rho = \gamma'' \rho_1, \rho_2 = \gamma \rho_1, \rho_3 = \gamma' \rho_1, \rho_4 = \gamma''' \rho_1$. The n-tuple ρ_1 is defined as in the smoothing theorem.

Definition: $H^l_* = \{ \xi \in H^l(X_0, \underline{F|X_0})$ such that there exists $U = U(0)$ in E^n with $\hat{\xi} \in H^l(\psi^{-1}(U), \mathbf{F})$ and $\hat{\xi} \mid X_0 = \xi \}$. Serre's theorem gives $\dim_{\mathbf{C}} H^l_* \leqslant \dim_{\mathbf{C}} H^l(X_0, \underline{F|X_0}) < \infty$. In the following discussion we are given $\hat{\mathfrak{b}}_1, \ldots, \hat{\mathfrak{b}}_r$ in $Z^l(\hat{\mathfrak{U}}'(\rho_4), \mathbf{F})$ such that $\hat{\mathfrak{b}}_1 \mid X_0, \ldots \hat{\mathfrak{b}}_r \mid X_0$ constitute a base of the complex vector space H^l_*. For this to be possible, ρ_4 has to be chosen small enough. Here $\hat{\mathfrak{U}}'$ is a Stein covering of $X(\rho_1)$ and defined as in the smoothing theorem. We also assume that we are given a sequence of measure coverings as there. Further we construct the sequence so that there are still sufficiently many measure coverings in between $\mathfrak{V}$ and $\mathfrak{U}$. These are denoted by $\mathfrak{U}^*_\nu$. We have $\mathfrak{U} \gg \mathfrak{U}^*_1 \gg \mathfrak{U}^*_2 \gg \ldots \gg \mathfrak{V}$. The n-tupel ρ_3 is also fixed from now on and K always denotes (possibly different) constants.

Approximation Lemma: Let $\varepsilon > 0$. Then we can find ρ_2 such that: If $\rho \leqslant \rho_2$ and $\hat{\xi} \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbf{F})$ with $\| \hat{\xi} \|_\rho < \infty$ (the norm is taken with respect to $\hat{\mathfrak{U}}^*_1(\rho)$), then there exist $a_1, \ldots a_r \in I(E^n(\rho))$ and $\hat{\eta} \in C^{l-1}(\mathfrak{V}(\rho), \mathbf{F})$ such that $\tilde{\xi} = \hat{\xi} - \sum_1^r a_i \hat{\mathfrak{b}}_i - \delta\hat{\eta}$ on $\mathfrak{V}(\rho)$. Here $\tilde{\xi} \in Z^l(\mathfrak{V}(\rho), \mathbf{F})$ and $\| \tilde{\xi} \|_\rho \leqslant \varepsilon \| \hat{\xi} \|_\rho$ and $\| a_\nu \|_\rho, \| \hat{\eta} \|_\rho \leqslant K \| \hat{\xi} \|_\rho$. K is a fixed constant.

Proof. We shall first prove some results which are needed later on. Let $S \in \Gamma(\hat{U}_{\iota_0 \ldots \iota_l}(\rho), \mathbf{F})$. Choose $\iota \in \{ \iota_0, \ldots, \iota_l \}$. Now $(U^{(1)*}_{\iota_0 \ldots \iota_l})_\iota \subset \hat{U}_{\iota_0 \ldots \iota_l}$ because $\mathfrak{U}^*_1 \ll \mathfrak{U}$. The operations are always defined with respect to ρ_1. We can now restrict S to $(U^{(1)*}_{\iota_0 \ldots \iota_l})_\iota(\rho)$. In the chart $\mathscr{W}_\iota$ we can write $S = \sum a_\nu (t/\rho)^\nu$. Here $a_\nu \in qI(U^{(1)*}_{\iota_0 \ldots \iota_l})$. Now the a_ν are extended constantly and we get elements $\hat{a}_\nu \in \Gamma((U^{(1)*}_{\iota_0 \ldots \iota_l})_\iota, \mathbf{F})$. Let us put $S_\nu = \hat{a}_\nu \mid \hat{U}^{(2)*}_{\iota_0 \ldots \iota_l}$. We claim that $\| S_\nu \|_{\rho_1} \leqslant K \| S \|_\rho$. For obviously $\| S \|_\rho \geqslant | a_\nu (U^{(1)*}_{\iota_0 \ldots \iota_l}) |$ and we can use the Theorem I to prove that $\| S_\nu \|_{\rho 1} \leqslant K \| \hat{a}_\nu \,|\, (U^{(1)*}_{\iota_0 \ldots \iota_l})_\iota (\rho_1) \|_\iota =$ $= K \,|\, a_\nu (U^{(1)*}_{\iota_0 \ldots \iota_l}) \,|\, \leqslant K \| S \|_\rho$. Q.E.D.

Let S_ν' be defined using some other $\iota' \in \{ \iota_0, \ldots \iota_l \}$. Then $S_\nu - S_\nu' \in$ $\in \Gamma (\hat{U}^{(2)*}_{\iota_0 \ldots \iota_l}, \mathbf{F})$. We claim that $\| S_\nu - S_\nu' \|_{\rho_4} \leqslant K \gamma''' \| S \|_\rho$.

Proof. Define $\alpha_s = \sum_{|\lambda| = s}^{\infty} a_\lambda (t/\rho)^\lambda$ and $\beta_s = \sum_{|\lambda|=0}^{s-1} a_\lambda (t/\rho)^\lambda$ over $(U^{(1)*}_{\iota_0 \ldots \iota_l})_\iota (\rho)$. We do the same for ι' respectively and obtain α_s' and β_s' over $(U^{(1)*}_{\iota_0 \ldots \iota_l})_{\iota'} (\rho)$. For the restrictions to $\hat{U}^{(2)*}_{\iota_0 \ldots \iota_l}$ we see that $\alpha_s - \alpha_s' =$ $-(\beta_s - \beta_s')$. Hence we get $\| \alpha_s - \alpha_s' \|_{\rho_4} \leqslant K (\gamma''')^s \| \alpha_s - \alpha_s' \|_{\rho_1} = K (\gamma''')^s \| \beta_s -$ $- \beta_s \|_{\rho_1} \leqslant K (\gamma''')^s \| \beta_s \|_{\rho_1} + K (\gamma''')^s \| \beta_s' \|_{\rho_1} \leqslant K (\gamma''')^s [\| \beta_s \|_{\rho_1}^* + \| \beta_s' \|_{\rho_1}^*] \leqslant$ $\leqslant K (\gamma''')^s (\gamma'')^{1-s} \| S \|_\rho$. Here the norms are defined with respect to $U^{(3)*}_{\iota_0 \ldots \iota_l}$ except $\| \; \|^*$ and $\| S \|_\rho$ which are defined with respect to $U^{(1)*}_{\iota_0 \ldots \iota_l}$. Now we look at the difference $(S_\nu - S_\nu') \, t^\nu / \rho^\nu$ on $(U^{(3)*}_{\iota_0 \ldots \iota_l})_\mu$ with $|\nu| = s$, $\mu \in \{ \iota_0, \ldots \iota_l \}$, and the power series development with respect to W_μ. There is one term of order s which is equal to the corresponding term of $\alpha_s - \alpha_s'$. Therefore its norm is $\leqslant K (\gamma''')^s . (\gamma'')^{1-s} \| S \|_\rho$. Moreover we have $\| S_\nu (t/\rho)^\nu - S_\nu' (t/\rho)^\nu \|_{\rho_1} \leqslant$ $\leqslant (\gamma'')^{-s} \cdot K \| S \|_\rho$ where the first norm is defined with respect to $U^{(3)*}_{\iota_0 \ldots \iota_l}$. For the sum $\sum$ of terms of higher order than s in the power series of $(S_\nu -$ $- S_\nu') \, t^\nu / \rho^\nu$ we therefore get: $\| \sum \|_{\rho_4} \leqslant (\gamma''')^{s+1} (\gamma'')^{-s} \cdot K \| S \|_\rho$. Hence we get $\| (S_\nu - S_\nu') \|_{\rho_4} \leqslant \gamma''' \cdot K \| S \|_\rho$. This proves our statement. We see that K is independent of ρ_4 and S. The number γ''' depends on ρ_4 only, so $\gamma''' \cdot K$ gets very small if we make ρ_4 very small.

Let $\hat{\xi} \in Z^l (\hat{\mathfrak{U}} (\rho), \mathbf{F})$ with $\hat{\xi} = \{ \hat{\xi}_{\iota_0 \ldots \iota_l} \}$. Choose $\iota = \iota (\iota_0, \ldots, \iota_l)$ as a function of the unordered $(l+1)$-tuple. We now fix $\iota_0, \ldots, \iota_l$ and write $S = \hat{\xi}_{\iota_0 \ldots \iota_l}$. We apply to S the method described above and obtain $\hat{\xi}^{(\nu)}_{\iota_0 \ldots \iota_l} =$ $= S_\nu$. We do this now for every $\iota_0, \ldots, \iota_l$ and consider $\hat{\xi}_{(\nu)} = \{ \hat{\xi}^{(\nu)}_{\iota_0 \ldots \iota_l} \}$ as an element of $C^l (\hat{\mathfrak{U}}_2^* (\rho_4), \mathbf{F})$. Of course $\hat{\xi}_{(\nu)}$ depends on the choice of $\iota = \iota (\iota_0 \ldots \iota_l)$ here. Now we see that $\| \hat{\xi}_{(\nu)} \|_{\rho 4} \leqslant \| \hat{\xi}_{(\nu)} \|_{\rho_1} \leqslant K \| \hat{\xi} \|_\rho$. We also wish to estimate $\delta \hat{\xi}_{(\nu)}$. Because $\hat{\xi} \in Z^l (\hat{\mathfrak{U}} (\rho), \mathbf{F})$ we can use the preliminary result on ι and ι' to obtain $\| \delta \hat{\xi}_{(\nu)} \|_{\rho 4} \leqslant K \gamma''' \| \hat{\xi} \|_\rho$.

We shall also need another result:

Induction Lemma: There exists $\hat{\eta}_\nu \in C^l(\mathfrak{U}_4^*(\rho_3), \mathbf{F})$ such that $\delta\hat{\eta}_\nu = \delta\hat{\xi}_{(\nu)}$ on $\hat{\mathfrak{U}}_4^*(\rho_3)$ and $\|\hat{\eta}_\nu\|_{\rho_3} \leqslant K \|\delta\hat{\xi}_{(\nu)}\|_{\rho_4}$.

Proof. The proof uses the assumption that $\psi_{(l+1)}(\mathbf{F})$ is coherent. Because the coherence of direct images is proved by downward induction on l, this assumption can be made. Moreover it is assumed that the main theorem is proved for dimension $l+1$ already. Let us now put $\alpha = \delta\hat{\xi}_{(\nu)} \in B^{l+1}(\hat{\mathfrak{U}}_2^*(\rho_4), \mathbf{F})$ and $\hat{\eta}_\nu = \beta \in C^l(\hat{\mathfrak{U}}_4^*(\rho_3), \mathbf{F})$. We have to prove the existence of β. We may assume that ρ_4 is so small that the main theorem is valid for $\rho \leqslant \rho_4$ in the case of dimension $l+1$. So there are cocycles $\omega_1, \ldots, \omega_r \in Z^{l+1}(\hat{\mathfrak{U}}(\rho_4), \mathbf{F})$ such that $\alpha = \sum C_\lambda \omega_\lambda + \delta\eta$, where $C_\lambda \in I(E^n(\rho_4))$ and $\eta \in C^l(\hat{\mathfrak{U}}_4^*(\rho_4), \mathbf{F})$. We have to assume that between $\hat{\mathfrak{U}}_4^*$ and $\mathfrak{U}_2^*$ there are very many measure coverings. The cross-sections $\psi_{(l+1)}(\omega_\lambda)$ give a homomorphism $r\mathcal{O} \to \psi_{(l+1)}(\mathbf{F})$ over $E^n(\rho_4)$. Because $\psi_{(l+1)}(\mathbf{F})$ is coherent the kernel $\mathcal{N}$ is coherent again. Over $E^n(\rho')$ with $\rho_3 < \rho' < \rho_4$ we find an epimorphism $p\mathcal{O} \to \mathcal{N}$. Denote by $n_1, \ldots, n_p$ the images of the unit cross-sections in $p\mathcal{O}$. Write $n_\lambda = (e_{\lambda 1}, \ldots, e_{\lambda r})$ as an r-tupel of holomorphic functions. The image of n_λ in $\Gamma(E^n(\rho'), \psi_{(l+1)}(\mathbf{F}))$ is $\psi_{(l+1)}\left(\sum_{\mu=1}^{r} e_{\lambda\mu}\,\omega_\mu\right)$ and zero. We may choose ρ_2 and then ρ_3 and ρ' very small. Then it follows that $\hat{n}_\lambda = \sum e_{\lambda\mu}\,\omega_\mu$ is a coboundary. If $\rho_3 < \rho'' < \rho'$ there are cochains $\eta_\lambda \in C^l(\hat{\mathfrak{U}}_4^*(\rho''), \mathbf{F})$ such that $\delta\eta_\lambda = \hat{n}_\lambda$. Now $(C_1, \ldots, C_r) \in \Gamma(E^n(\rho_4), \mathcal{N})$. By the methods of sheaf theory we can lift this cross-section to $p\mathcal{O}$. Using a " Banach open mapping theorem " we see that the map $\Gamma(E^n(\rho'), p\mathcal{O}) \to \Gamma(E^n(\rho'), \mathcal{N})$ is open. This means here that we can find holomorphic functions a_λ over $E^n(\rho_3)$ such that $C_\mu = \sum a_\lambda e_{\lambda\mu}$ and $\|a_\lambda\|_{\rho_3} \leqslant K \max_\mu \|C_\mu\|_{\rho'} \leqslant K \max_\mu \|C_\mu\|_{\rho_4}$. We get $\sum C_\mu\,\omega_\mu = \sum a_\lambda\, e_{\lambda\mu}\omega_\mu = \sum a_\lambda \hat{n}_\lambda = \delta(\sum a_\lambda \eta_\lambda)$. This leads to $\alpha \mid C^{l+1}(\hat{\mathfrak{U}}_4^*(\rho_3)) = \delta(\eta + \sum a_\lambda \eta_\lambda)$. The estimates required obviously hold. Q.E.D.

Let us now put $\hat{\xi}_{(\nu)}^* = \hat{\xi}_{(\nu)} - \hat{\eta}_\nu \in Z^l(\hat{\mathfrak{U}}_4(\rho_3), \mathbf{F})$. We can write $\hat{\xi}_{(\nu)}^* \mid X_0 = \sum a_{\nu\lambda}\,\hat{\mathfrak{b}}_\lambda \mid X_0 + \delta\gamma_\nu$ over $\mathfrak{U}_6^*$. Here $a_{\nu\lambda}$ are complex numbers and $\gamma_\nu \in C^{l-1}(\mathfrak{U}_6^*, F \mid X_0)$. Cartan's theorem and the result after that give the estimates $|a_{\nu\lambda}| \leqslant K \|\hat{\xi}_{(\nu)}^*\|_{\rho_3} \leqslant K \|\hat{\xi}\|_\rho$ and $\|\hat{\gamma}_\nu\|_{\rho_3} \leqslant K \|\hat{\xi}_{(\nu)}^*\|_{\rho_3} \leqslant K \|\hat{\xi}\|_\rho$. Here $\hat{\gamma}_\nu \in C^{l-1}(\hat{\mathfrak{U}}_7^*(\rho_3), \mathbf{F})$ has been obtained by a constant extension of γ_ν. Let us now put $\hat{\xi}^{(1)}_{(\nu)} = \hat{\xi}^{*}_{(\nu)} - \sum a_{\nu\lambda} \hat{\mathfrak{b}}_\lambda - \delta\hat{\gamma}_\nu$. Here $\hat{\xi}^{(1)}_{(\nu)} \in$ $\in C^l(\hat{\mathfrak{U}}^*_7(\rho_3), \mathbf{F})$. Using the previous estimates and the fact that the $\hat{\mathfrak{b}}_\lambda$ are finite we find that $\| \hat{\xi}^{(1)}_{(\nu)} \|_{\rho_3} \leqslant K \| \hat{\xi}_{(\nu)} \|_{\rho_4} \leqslant K \| \hat{\xi} \|_\rho$.

Now we also have $\hat{\xi}^{(1)}_{(\nu)} \mid X_0 = 0$. It follows that

$$\| \hat{\xi}^{(1)}_{(\nu)} \|_\rho \leqslant \gamma/\gamma' \| \hat{\xi}^{(1)}_{(\nu)} \|_{\rho_3} \leqslant \gamma/\gamma' \cdot K \| \hat{\xi} \|_\rho .$$

Finally we put in $\hat{\mathfrak{U}}^*_9(\rho)$:

$$\hat{\xi}^{(1)} = \Sigma \hat{\xi}^{(1)}_{(\nu)} (t/\rho)^\nu =$$

$$= \Sigma \hat{\xi}_{(\nu)} (t/\rho)^\nu - \Sigma \hat{\eta}_\nu (t/\rho)^\nu - \Sigma a_{\nu\lambda} (t/\rho)^\nu \hat{\mathfrak{b}}_\lambda - \delta(\Sigma \hat{\gamma}_\nu (t/\rho)^\nu)$$

$$= \hat{\xi} - \hat{\eta} - \Sigma a_\lambda \hat{\mathfrak{b}}_\lambda - \delta\hat{\gamma} .$$

Using the fact that the sum of the absolute values of the coefficients in the power series expansion of $\hat{\xi}^{(1)}_{(\nu)}$ by (t/ρ) is smaller than $\gamma/\gamma' \cdot K \| \hat{\xi} \|_\rho$ and that with respect to $\hat{\eta}_\nu$ is smaller than $\gamma''' \cdot K \| \hat{\xi} \|_\rho$ we find: $\| \hat{\xi}^{(1)} \|_\rho \leqslant$ $\leqslant \gamma/\gamma' \cdot K \| \hat{\xi} \|_\rho$ and $\| \hat{\eta} \|_\rho \leqslant \gamma''' \cdot K \| \hat{\xi} \|_\rho$ and $\| a_\lambda \|_\rho \leqslant K \| \hat{\xi} \|_\rho$. We take the restriction to $\hat{\mathfrak{B}}(\rho)$ and now $\tilde{\xi} = \hat{\xi}^{(1)} - \hat{\eta} \in Z^l(\hat{\mathfrak{B}}(\rho), \mathbf{F})$ is the desired element. Of course we have to choose ρ_4 and then ρ_2 small enough, for example let $\gamma''' < \varepsilon/2K$ and $\gamma \leqslant \varepsilon\gamma'/2K$.

Main Theorem

There exists ρ_2 and a constant K such that if $\rho \leqslant \rho_2$ and $\hat{\xi} \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbf{F})$ with $\| \hat{\xi} \|_\rho < \infty$ then we can find $a_1, \ldots, a_r \in I(E^n(\rho))$ and $\hat{\eta} \in$ $C^{l-1}(\hat{\mathfrak{B}}(\rho), \mathbf{F})$ such that $\hat{\xi} = \sum a_\lambda \hat{\mathfrak{b}}_\lambda + \delta\hat{\eta}$ on $\hat{\mathfrak{B}}(\rho)$ with $\| \hat{\eta} \|_\rho$ and $\| a_\nu \|_\rho \leqslant$ $\leqslant K \| \hat{\xi} \|_\rho$.

Proof. We have one constant K from the smoothing theorem. Now we find ρ_2 with an ε in the Approximation Lemma such that $\varepsilon \cdot K < 1/2$. We shall use this ρ_2 and prove the theorem here. We are given $\hat{\xi}_0 = \hat{\xi} \in$ $Z^l(\hat{\mathfrak{U}}(\rho), \mathbf{F})$ with $\| \hat{\xi} \|_\rho < \infty$. The Approximation Lemma gives $\tilde{\xi}_1 =$