1. Introduction

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ONE-SIDED ANALOGUES OF KARAMATA's REGULAR VARIATION *)

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To the memory of J. Karamata

1. INTRODUCTION

The monotone functions studied in this paper are assumed to be defined on $(0, \infty)$, and to be non-negative and right continuous. Point functions are introduced for notational convenience only, but we are really concerned with the associated measure which attributes value $U(x)$ to the interval $[0, x]$ when U increases, and to (x, ∞) when U decreases to zero.

Karamata's original theory of regularly varying functions has been generalized in chapter VIII of [1] to measures. A monotone function U *is said to vary regularly at infinity with exponent* α *if*

$$(1.1) \qquad \lim_{t \to \infty} \frac{U(tx)}{U(t)} = x^{\alpha}$$

for some α and each $x > 0$. At first glance this definition appears to be artificial, but it is motivated by the fact that if the limit on the left exists, it is necessarily of the form x^{α}. Accordingly *U varies regularly at infinity if, and only if, as $t \to \infty$ the measures $U(t\mathrm{d}x)/U(t)$ converge to a finite measure* in every finite interval. (Here, and in the following, convergence of measures is taken in the usual weak sense.) The function U varies regularly at the origin if $U(t^{-1})$ varies regularly at infinity. From now on we omit the qualification " at infinity ", and it will be tacitly understood that in our passages to the limit the variable tends to ∞.

With an arbitrary measure U on $(0, \infty)$ we may associate the truncated moment functions

$$(1.2) \qquad U_p(x) = \int_x^{\infty} y^{-p}\, U(dy)$$

* Work connected with a Project for research in probability theory at Princeton University, supported by the Army Research Office.

and

$$(1.3) \qquad Z_q(x) = \int_0^x y^q \, U(dy)$$

defined whenever the integrals converge. These functions define new measures and they satisfy obvious identities such as

$$(1.4) \qquad U_p(x) = \int_x^\infty y^{-p-q} Z_q(dy) .$$

For convenience of notation and exposition we shall from now on use $U = Z_0$ as representative of the whole family $\{Z_q\}$ and formulate all theorems in terms of U and U_p. Various relations between the diverse Z_q will be implicit in our theorems. As a last piece of notation we introduce the frequently occurring function

$$(1.5) \qquad R_U(t) = \frac{t^p \, U_p(t)}{U(t)} .$$

The notion of regular variation was introduced, and achieved its greatest success, in connection with Tauberian theorems. In recent years more attention was paid to hitherto little known relations derived by Karamata in [3] and connecting the asymptotic behavior of the various functions Z_q and U_p. The basic theorems may be summarized as follows.

(i) *Let U vary regularly with exponent $\alpha \geqslant 0$. Then U_p exists for $p > \alpha$ (but for no $p < \alpha$). Furthermore*

$$(1.6) \qquad \lim_{t \to \infty} R_U(t) = r$$

exists, and $r = \dfrac{\alpha}{p-\alpha} < \infty$. *If $r > 0$ then U_p varies regularly with exponent $\alpha - p$.*

(ii) *Let U be such that U_p exists for some fixed $p > 0$ and varies regularly with some exponent $q \leqslant 0$. Then the limit (1.6) exists, $0 < r \leqslant \infty$. If $r < \infty$ then U varies regularly with exponent $p+q$.*

(iii) *Let the limit r exist. If $0 < r < \infty$ then both U and U_p vary regularly with exponents*

$$(1.7) \qquad \alpha = p \frac{r}{r+1} \text{ and } \alpha - p = -p \frac{1}{r+1}$$

respectively ; hence

$$U_p(t) \sim rt^{-p}\, U(t)\,. \tag{1.8}$$

If $r = 0$ *then* U *varies slowly* ($\alpha = 0$) *and* $U_p(t) = o(t^{-p}\, U(t))$. *If* $r = \infty$ *then* U_p *varies slowly and* $U(\mathrm{t}) = o(t^p\, U_p(t))$.

Thus except when either U or U_p varies slowly regular variation is tied to a relation of the form (1.8) and the existence of the limit (1.6) characterizes regular variation.

Karamata considered only measures defined by densities. Simplified proofs and extensions of his results can be found in [1]. In this book it was shown that the measure theoretic version of Karamata's relations introduces coherence and unity in the theory of domains of attraction, and that it leads to a substantial simplification of this theory. (In such connections U is usually the truncated second moment of a probability distribution, and U_2 is then the tail sum of this distribution.)

In [2] it turned out that various compactness arguments and local limit theorems in probability do not depend on the full strength of regular variation, but only on a one sided version of it.

We now proceed to describe this generalization and to show that Karamata's relations carry over to a surprising extent. In section 4 we discuss inequalities going in the opposite direction.

In section 5 we turn to *ratio limit theorems*. Roughly speaking, we show that if two monotone functions U and V stand in the relation $V = UL$ with L slowly varying, then also $V_p \sim U_p\, L$. For regularly varying function this is implicit in Karamata's relation (1.8), but it is surprising that dominated variation should suffice for the conclusion. Furthermore, we obtain a necessary and sufficient condition for the ratio V/U to be slowly varying. In section 6 these results are reformulated in the form of a *Tauberian ratio limit theorem*.

To illustrate the way in which dominated variation occurs naturally in probabilistic contexts we discuss in section 7 the asymptotic behavior of the *tails of infinitely divisible distributions*. This section is independent of the Karamata relations and may be read directly after section 2.