

5. Ratio limit theorems

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **15 (1969)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Again, if it is known that R_U is bounded away from 0 then (4.5) shows that (4.2) implies (4.1).

We have thus proved the

COROLLARY. *If U is of dominated variation with exponent $\gamma < p$ then (4.1) implies (4.2). Similarly, if U_p is of dominated variation with exponent $-q$ where $q < p$, then (4.2) entails (4.1). (In each case both functions are of dominated variation.)*

5. RATIO LIMIT THEOREMS

Let U and V be non-decreasing unbounded functions, and suppose that L is slowly varying (= regularly varying with exponent 0).

DEFINITION. *We shall say that U and V are L -equivalent and write*

$$(5.1) \quad V \leftrightarrow UL$$

if the ratio UL/V tends to 1 at all points of continuity.

More precisely, it is required that for each $\varepsilon > 0$ and fixed $\lambda > 1$

$$(5.2) \quad (1 - \varepsilon) L(t) U(t/\lambda) \leq V(t) \leq (1 + \varepsilon) L(t) U(t\lambda)$$

for all t sufficiently large.

THEOREM 4. *Let U be of dominated variation. In order that there exist a slowly varying function L such that (5.1) holds it is necessary and sufficient that*

$$(5.3) \quad R_U(t) - R_V(t) \rightarrow 0 \quad \text{boundedly.}$$

Needless to say, R_V and \mathcal{J}_V are defined by analogy with R_U in (1.5) and \mathcal{J}_U in (3.2).

PROOF. (a) *Necessity.* Assume (5.1) and suppose that U satisfies the basic inequality (2.2). Obviously the slow variation of L implies that for t sufficiently large and all $x > 1$

$$(5.4) \quad \frac{V(tx)}{V(t)} < C' x^{\gamma'}$$

for any pair of constants $C' > C$ and $\gamma' > \gamma$. Thus V is of dominated variation, and since $p > \gamma$ the function V_p exists.

Let $t_n \rightarrow \infty$ in such a way that the measures associated with $U(t_n \cdot)/U(t_n)$ tend (in finite intervals) to a limit measure m . The relation (5.1) implies obviously that the measures associated with $V(t_n \cdot)/V(t_n)$ tend to the same limit m . Thus when t runs through $\{t_n\}$ we have for fixed $x > 1$

$$(5.5) \quad \frac{U_p(t) - U_p(tx)}{U(t)t^{-p}} = \int_1^x y^{-p} \frac{U(tdy)}{U(t)} \rightarrow \int_1^x y^{-p} m(dy),$$

and the same relation holds with U replaced by V . But (5.4) implies that this passage to the limit is uniform as $x \rightarrow \infty$; it remains valid also for $x = \infty$ with the right side being finite. We have thus shown that $R_U(t_n) - R_V(t_n) \rightarrow 0$. But the t_n may be picked as elements of an arbitrarily prescribed sequence, and so the limit relation in (5.3) holds pointwise for an arbitrary approach $t \rightarrow \infty$. Now we know that the dominated variation of U and V implies the boundedness of both R_U and R_V , and the condition (5.3) holds true.

(b) *Sufficiency.* The variation of U being dominated, R_U remains bounded and so (5.3) implies the boundedness of R_V and hence the dominated variation of V . The calculation of part (ii) in section 3 show that

$$(5.6) \quad \frac{s^{-p-1} U(s)}{\mathcal{I}_U(s)} - \frac{s^{-p-1} V(s)}{\mathcal{I}_V(s)} = \frac{p}{t} \left[\frac{1}{1 + R_U(s)} - \frac{1}{1 + R_V(s)} \right].$$

The expression within brackets is in absolute value bounded by $|R_U(s) - R_V(s)|$, and therefore tends to 0 boundedly. Integrating between t and $tx > t$ we conclude therefore that

$$(5.7) \quad \log \frac{\mathcal{I}_V(t)}{\mathcal{I}_U(tx)} \cdot \frac{\mathcal{I}_V(tx)}{\mathcal{I}_U(t)} \rightarrow 0.$$

In other words, the ratio $\mathcal{I}_U/\mathcal{I}_V$ varies slowly, and therefore we can put

$$(5.8) \quad \mathcal{I}_V(t) = L(t) \mathcal{I}_U(t)$$

where L varies slowly.

We now recall the inequality (3.14) which implies that to each $\lambda > 1$ there exists an $\eta < 1$ such that

$$(5.9) \quad \mathcal{I}_U(\lambda t) < \eta \mathcal{I}_U(t)$$

for all t sufficiently large. From (5.8) we conclude therefore that

$$(5.10) \quad \begin{aligned} & \lim \frac{\mathcal{I}_V(t) - \mathcal{I}_V(\lambda t)}{[\mathcal{I}_U(t) - \mathcal{I}_U(\lambda t)] L(t)} = \\ & = \lim \frac{L(t) \mathcal{I}_U(t) - L(\lambda t) \mathcal{I}_U(\lambda t)}{L(t) \mathcal{I}_U(t) - L(t) \mathcal{I}_U(\lambda t)} = 1. \end{aligned}$$

But the fraction on the left lies between

$$\frac{V(\lambda t)}{U(t) L(t)} \quad \text{and} \quad \frac{V(t)}{U(\lambda t) L(t)}$$

and so (5.1) is true.

6. APPLICATION TO TAUBERIAN THEOREMS

If the measure U varies regularly at infinity, then its Laplace transform ω varies regularly at the origin. More precisely, Karamata's now classical Tauberian theorem states that for any $\alpha \geq 0$ and slowly varying function L the two relations

$$(6.1) \quad U(x) \sim x^\alpha L(x) \quad \omega(\lambda) \sim \Gamma(\alpha + 1) \lambda^{-\alpha} L(\lambda^{-1})$$

imply each other; here $x \rightarrow \infty$ but $\lambda \rightarrow 0$. [The sign \sim indicates that the ratio of the two sides tends to 1.] For an example of a probabilistic application suppose that

$$(6.2) \quad U(x) = \int_0^x y^p F(dy)$$

is the truncated p^{th} moment of a probability distribution F on the positive half axis. For simplicity let p stand for a positive integer. Then $U_p(x) = 1 - F(x)$ and $\omega = (-1)^p \phi^{(p)}$ where ϕ is the Laplace-Stieltjes transform of F . If ω varies regularly in accordance with (6.1) then Karamata's relation (1.8) implies that

$$(6.3a) \quad 1 - F(x) \sim \frac{\alpha}{p - \alpha} x^{\alpha-p} L(x) \quad \text{when} \quad \alpha < p$$

$$(6.3b) \quad 1 - F(x) = o(x^\alpha L(x)) \quad \text{when} \quad \alpha = p.$$

(Note that necessarily $0 \leq \alpha \leq p$ because the measure F is finite.) In other words, the behavior at the origin of the derivatives of the Laplace transform determines the behavior of the tail $1 - F(x)$, and vice versa.