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ASYMPTOTIC BEHAVIOUR OF A CLASS OF DETERMINANTS

by Mark KAC

To the memory of J. Karamata

1° Let $f(\theta)$, $0<\theta<1$, be a function which in addition to certain regularity conditions, which will be specified later on, is bounded from below by a number larger than 2. In other words,

$$f(\theta) > m > 2\,, \qquad 0 < \theta < 1\,. \tag{1.1}$$

Consider the determinant

$$D_n(f) = \begin{vmatrix} f(\frac{1}{n}), & -1, & 0, & \dots & - & . & 0 \\ -1, & f(\frac{2}{n}), & -1, & \dots & & . & 0 \\ 0, & -1, & f(\frac{3}{n}), & -1, & \dots & & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & \dots & & & -1, & f(\frac{n}{n}) \end{vmatrix} \tag{1.2}$$

i.e. the determinant of the matrix

$$a_{ij} = f(\tfrac{i}{n})\,\delta_{ij} - \delta_{i,j-1} - \delta_{i,j+1} \tag{1.3}$$

where δ, as usual, denotes the Kronecker delta.

If $f(\theta)$ is Riemann integrable it can be shown that

$$\lim_{n\to\infty} D_n^{\frac{1}{n}}(f) =$$

$$\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega \int_0^1 \log\left(f(\theta) - 2\cos\omega\right) d\theta\right\} = G(f)\,. \tag{1.4}$$

This is a special case of a more general result of Kac, Murdock and Szegô [1953].

Assuming that $f(\theta)$ is twice differentiable, with a bounded second derivative, one can go a good deal farther.

In fact, one can determine

$$\lim_{n\to\infty}\frac{D_n(f)}{G^n(f)} \tag{1.5}$$

with $G(f)$ as defined in (1.4).

Although the result is again a special case of a more general result obtained by Mejlbo and Schmidt [1962] the method which we shall use is quite different and of some interest in itself. The method has also the advantage of explaining the role of smoothness conditions imposed on f.

2° We begin with the elementary formula

(2.1) $$D_n^{-\frac{1}{2}}(f) =$$

$$\left(\frac{1}{(\sqrt{\pi})^n}\right)^n \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left\{-\sum_1^n f\left(\frac{k}{n}\right)x_k^2+2\sum_1^{n-1}x_k x_{k+1}\right\}dx_1\,dx_2\ldots dx_n$$

and note that setting

$$K\left(x,y;\frac{l}{n}\right)=\frac{1}{\sqrt{\pi}}\exp\left\{-\frac{1}{2}f\left(\frac{l}{n}\right)x^2+2xy-\frac{1}{2}f\left(\frac{l+1}{n}\right)y^2\right\} \tag{2.2}$$

we can rewrite (2.1) in the form

(2.3) $$D_n^{-\frac{1}{2}}(f) =$$

$$=\frac{1}{(\sqrt{\pi})^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}f\left(\frac{1}{n}\right)x_1^2\right\}K\left(x_1,x_2;\frac{1}{n}\right)K\left(x_2,x_3;\frac{2}{n}\right)\ldots$$

$$\ldots K\left(x_{n-1},x_n;\frac{n}{n}\right)\exp\left\{-\frac{1}{2}f\left(\frac{n}{n}\right)x_n^2\right\}dx_1\,dx_2\ldots dx_n\,.$$

The kernels $K(x,y;\frac{l}{n})$ are not symmetric and we must replace them by symmetric ones without introducing too large an error.

This is done by noting that

$$f\left(\frac{l}{n}\right)=\frac{1}{2}f\left(\frac{l-1}{n}\right)+\frac{1}{4n}f'\left(\frac{l-1}{n}\right)+\frac{1}{2}f\left(\frac{l}{n}\right)+\frac{1}{4n}f'\left(\frac{l}{n}\right)+0\left(\frac{1}{n^2}\right) \tag{2.4}$$

where the error $0(1/n^2)$ is actually less than M/n^2 where M is a bound for $f''(\theta)$ in $(0, 1)$.

It follows that

$$(2.5)\qquad \Bigg| \sum_{l=1}^{n} f\left(\frac{l}{n}\right) x_l^2 - \left\{\left(\frac{1}{2} f(0) + \frac{1}{4n} f'(0)\right) x_1^2 + \right.$$

$$+ \sum_{l=1}^{n-1} \left[\left(\frac{1}{2} f\left(\frac{l}{n}\right) + \frac{1}{4n} f'\left(\frac{l}{n}\right)\right) x_l^2 + \left(\frac{1}{2} f\left(\frac{l}{n}\right) + \frac{1}{4n} f'\left(\frac{l}{n}\right)\right) x_{l+1}^2\right]$$

$$\left. + \left(\frac{1}{2} f(1) + \frac{1}{4n} f'(1)\right) x_n^2 \right\} \Bigg| < \frac{M}{n^2} \sum_{1}^{n} x_l^2$$

and this suggests that we introduce the symmetric kernels

$$(2.6)\qquad \tilde{K}_\alpha\left(x, y; \frac{l}{n}\right) = \frac{1}{\sqrt{\pi}} \exp\left\{ - \left(\frac{1}{2} f\left(\frac{l}{n}\right) + \frac{1}{4n} f'\left(\frac{l}{n}\right) + \frac{\alpha}{2n^2}\right) x^2 + 2xy \right.$$

$$\left. - \left(\frac{1}{2} f\left(\frac{l}{n}\right) + \frac{1}{4n} f'\left(\frac{l}{n}\right) + \frac{\alpha}{2n^2}\right) y^2 \right\}.$$

It should now be clear from (2.3) and (2.5) that $D_n^{-\frac{1}{2}}(f)$ is contained between the value of the integral

$$(2.7)\qquad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left\{ -\left(\frac{1}{2} f(0) + \frac{1}{4n} f'(0)\right) x_1^2 \right\} \prod_{l=1}^{n-1} K_\alpha\left(x_l, x_{l+1}; \frac{l}{n}\right)$$

$$\exp\left\{ - \left(\frac{1}{2} f(1) + \frac{1}{4n} f'(1)\right) x_n^2 \right\} dx_1 \ldots dx_n$$

for $\alpha = -M$ and its value for $\alpha = M$.

3° It will turn out that the asymptotic behaviour of (2.7), to the required accuracy, does not depend on α so that we may as well set $\alpha = 0$.

The kernels $K_0(x, y; l/n)$ do not commute. However they " almost commute " in the following sense:

Let $\lambda_1(l/n), \lambda_2(l/n), \ldots$ be the eigenvalues of $K_0(x, y; l/n)$ in the decreasing order and let

$$\varphi_1(x; l/n), \quad \varphi_2(x; l/n), \ldots$$

be the corresponding normalized eigenfunctions.

Then

$$(3.1)\qquad \int_{-\infty}^{\infty} \varphi_i\left(x; \frac{l}{n}\right) \varphi_j\left(x; \frac{l+1}{n}\right) dx = \delta_{ij} + 0\left(\frac{1}{n^2}\right).$$

This follows by a perturbation argument but can also be obtained directly since the eigenfunctions and eigenvalues of our kernels can be determined explicitly.

I shall now ask the reader to believe me that whenever we have (3.1) and whenever $\lambda_1(l/n) - \lambda_2(l/n)$ is bounded away from 0 (uniform non-degeneracy) the integral (2.7) divided by

(3.2)
$$\prod_{l=1}^{n-1} \lambda_1 \left(\frac{l}{n}\right)$$

approaches

(3.3)
$$\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} \varphi_1(x; 0) \exp\left\{-\frac{1}{2} f(0) x^2\right\} dx \int_{-\infty}^{\infty} \varphi_1(x; 1) \exp\left\{-\frac{1}{2} f(1) x^2\right\} dx .$$

In ogher words, the asymptotic behaviour of the integral (2.7) is the same *as if* the kernels $K_0 (x, y; l/n)$ commuted. The details of the proof are tedious but intuitively the assertions should be nearly obvious.

It remains now to calculate $\lambda_1 (l/n)$ and $\varphi_1 (x; l/n)$.

The kernels K_0 are of the form

(3.4)
$$\frac{1}{\sqrt{\pi}} \exp\left\{-\frac{a}{2} x^2 + 2xy - \frac{a}{2} y^2\right\}$$

with $a > 2$ and it is easy to convince oneself that the principal eigenfunction is of the form

(3.5)
$$\frac{\sqrt[4]{b}}{\sqrt[4]{\pi}} \exp\left\{-\frac{b}{2} y^2\right\}$$

A simple calculation then gives

(3.6)
$$b = \sqrt{a^2 - 4}$$

and for the maximum eigenvalue $\lambda_1 (a)$ the formula

(3.7)
$$\lambda_1 (a) = \left(\frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - 1}\right)^{-\frac{1}{2}} .$$

Setting

(3.8)
$$a_l = f\left(\frac{l}{n}\right) + \frac{1}{2n} f'\left(\frac{l}{n}\right)$$

we finally obtain that

$$(3.9)\qquad \lim_{n\to\infty} \frac{D_n(f)}{\prod_{l=1}^{n-1}\left(\frac{a_l}{2}+\sqrt{\left(\frac{a_l}{2}\right)^2-1}\right)} = \frac{f(0)+\sqrt{f^2(0)-4}}{2\sqrt[4]{f^2(0)-4}}\cdot$$

$$\frac{f(1)+\sqrt{f^2(1)-4}}{2\sqrt[4]{f^2(1)-4}}.$$

It remains to determine the asymptotic behavior of the product

$$(3.10)\qquad \prod_{l=1}^{n-1}\left(\frac{a_l}{2}+\sqrt{\left(\frac{a_l}{2}\right)^2-1}\right) = \exp\sum_{l=1}^{n-1}\log\left(\frac{a_l}{2}+\sqrt{\left(\frac{a_l}{2}\right)^2-1}\right).$$

Neglecting terms of order $1/n^2$ and higher we find easily that

$$\log\left(\frac{a_l}{2}+\sqrt{\left(\frac{a_l}{2}\right)^2-1}\right) = \log\frac{f\left(\frac{l}{n}\right)+\sqrt{f^2\left(\frac{l}{n}\right)-4}}{2} + \frac{1}{2n}\frac{f'\left(\frac{l}{n}\right)}{\sqrt{f^2\left(\frac{l}{n}\right)-4}}$$

and hence

$$\sum_{l-1}^{n-1}\log\left(\frac{a_l}{2}+\sqrt{\left(\frac{a_l}{2}\right)^2-1}\right) =$$

$$(3.11)\qquad \sum_{l=1}^{n-1}\log\frac{f\left(\frac{l}{n}\right)+\sqrt{f^2\left(\frac{l}{n}\right)-4}}{2} + \frac{1}{2}\int_0^1\frac{f'(\theta)}{\sqrt{f^2(\theta)-4}}\,d\theta + 0\left(\frac{1}{n}\right).$$

Using the well known formula

$$(3.12)\qquad \sum_{l=1}^{n-1} g\left(\frac{l}{n}\right) = n\int_0^1 g(\theta)\,d\theta - \frac{g(0)+g(1)}{2} + 0\left(\frac{1}{n}\right)$$

for

$$(3.13)\qquad g(\theta) = \log\frac{f(\theta)+\sqrt{f^2(\theta)-4}}{2}$$

and noting that

$$\int_0^1\frac{f'(\Theta)}{\sqrt{f^2(\theta)-4}}\,d\theta = \log\frac{f(1)+\sqrt{f^2(1)-4}}{2} - \log\frac{f(0)+\sqrt{f^2(0)-4}}{2}$$

we obtain

$$\sum_{l=1}^{n-1} \log\left(\frac{a_l}{2} + \sqrt{\left(\frac{a_l}{2}\right)^2 - 1}\right) =$$

(3.14)

$$n \int_0^1 \log \frac{f(\theta) + \sqrt{f^2(\theta) - 4}}{2} \, d\theta - \log \frac{f(0) + \sqrt{f^2(0) - 4}}{2} + 0\left(\frac{1}{n}\right).$$

Combining (3.14) with (3.9) and noting that

$$\log G(f) = \frac{1}{2\pi} \int_0^1 d\theta \int_{-\pi}^{\pi} \log(f(\theta) - 2\cos\omega)\, d\omega =$$

$$\int_0^1 \log \frac{f(\theta) + \sqrt{f^2(\theta) - 4}}{2} \, d\theta$$

we are led to the final result that

$$\lim_{n\to\infty} \frac{D_n(f)}{G^n(f)} = \frac{f(1) + \sqrt{f^2(1) - 4}}{\sqrt[4]{f^2(0) - 4}\,\sqrt[4]{f^2(1) - 4}}. \tag{3.15}$$

4° If we set

$$\log(f(\theta) - 2\cos\omega) = \sum_{-\infty}^{\infty} h_\nu(\theta) e^{i\nu\omega} \tag{4.1}$$

the right hand side of (3.15) can be shown to be equal to

$$\exp\left\{\frac{1}{2}(h_0(1) - h_0(0))\right\} \exp\left\{\frac{1}{2}\sum_1^\infty \nu h_\nu^2(0) + \frac{1}{2}\sum_1^\infty \nu h_\nu^2(1)\right\}. \tag{4.2}$$

This is in apparent disagreement with the formula of Mejlbo and Schmidt since in their formula $h_0(1) - h_0(0)$ is replaced by $h_0(0) - h_0(1)$. We would get complete agreement if instead of our determinant (1.2) we would consider a modified determinant with $f(0), f(\frac{1}{n}), f(\frac{2}{n}), \ldots, f(\frac{n-1}{n})$ on the main diagonal. So accurate is formula (3.15) that it is sensitive to so slight a modification!

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