§ 4. Deduction of boundedness principles

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(2) Local convexity is needed in the proof of 3.1 since otherwise (2.2″), i.e., the boundedness of $S = \{x_n : n \in N\}$ in E, does not guarantee the existence of any continuous or bounded linear map T from $l^1(N)$ into E such that S is contained in the T-image of a bounded subset of $l^1(N)$. For it is plain that such a T can exist, only if the convex envelope S' of S is bounded in E. On the other hand, it is not difficult to verify that any first countable linear topological space E, in which the convex envelope of every bounded set (or of the range of every sequence converging to zero in E) is bounded, is necessarily locally convex.

(3) Naturally, local convexity of E may be dropped from the hypotheses of 3.1, if one assumes in place of (2.2″) that the convex envelope of $\{x_n : n \in N\}$ is a bounded subset of E.

§ 4. *Deduction of boundedness principles*

4.1 THEOREM. Suppose that E is a sequentially complete locally convex space and that P is a set of bounded gauges on E. If $f^*(x) = \sup\{f(x) : f \in P\} < \infty$ for every $x \in E$, then f^* is bounded.

PROOF. Suppose the contrary, that is, that $f^*(x) < \infty$ for every $x \in E$ and yet there exists a bounded subset B of E on which f^* is unbounded. Then we can choose $x_n \in B$, $f_n \in P$ such that $f_n(x_n) > n$ for every $n \in N$. Then (2.1), (2.2″) and (2.3) are satisfied; hence, by 3.1, there exists $x \in E$ such that $f^*(x) = \infty$, which is the required contradiction.

4.2 REMARKS. (1) If we assume also that E is infrabarrelled and that each $f \in P$ is continuous, it follows that f^* is continuous, that is, that P is equicontinuous if it is pointwise bounded; cf. [2], pp. 47, 480-81. For, if V denotes the interval $[-\varepsilon, \varepsilon]$, where $\varepsilon > 0$, then

$$f^{*-1}(V) = \bigcap \{f^{-1}(V) : f \in P\}$$

is closed, convex and balanced and absorbs bounded sets in E. Since E is infrabarrelled, $f^{*-1}(V)$ is therefore a neighbourhood of the origin in E and thus f^* is continuous, as asserted.

(2) If one drops the hypothesis that E be locally convex (the remaining assumptions of Theorem 4.1 remaining intact), the substance of Remark 3.3 (3) shows that one may still conclude that $f^*(B)$ is bounded whenever B is a subset of E whose convex envelope in E is bounded.

However, even assuming that E is first countable and complete, one can in general no longer conclude that f^* is bounded (i.e., that $f^*(A)$ is bounded for every bounded subset A of E) whenever it is finite-valued. Counter-examples are easily given in the case of the familiar spaces $E = l^p(N)$ with $p \in (0, 1)$.

Part 2: Applications to Multipliers

§ 5. (p, q)-*multipliers which are not measures*

5.1 Introduction. In this section and the following one we will use the substance of § 3 to prove several apparently new properties of (p, q)-multipliers. Let G be a locally compact group [all topological groups will be assumed to be Hausdorff and, in this section, will be multiplicatively written with identity e]. Denote by $L^p(G)$, where $1 \leqq p \leqq \infty$, the usual Lebesgue space formed with a fixed left Haar measure λ_G on G; and by $C_c(G)$ the space of continuous complex-valued functions on G with compact supports.

For $a \in G$, define the left translation operator τ_a and the right translation operator ρ_a by

$$\tau_a g(x) = g(a^{-1} x) \quad \text{and} \quad \rho_a g(x) = g(xa^{-1});$$

respectively. A linear operator T from $C_c(G)$ into $L^q(G)$ is said to be a *(left)* (p, q)*-multiplier* if and only if

(i) T is continuous from $C_c(G)$, equipped with the norm induced by $L^p(G)$, into $L^q(G)$; and

(ii) T commutes with left translations, that is $T\tau_a = \tau_a T$ for all $a \in G$.

A *right* (p, q)*-multiplier* is defined in a similar manner with (ii) replaced by

(ii′) $T\rho_a = \rho_a T$ for all $a \in G$.

Let $L_p^q(G)$ denote the Banach space of (p, q)-multipliers equipped with the customary norm, denoted by $\|\cdot\|_{p,q}$, of continuous linear operators from a subspace of $L^p(G)$ into $L^q(G)$. That is, for each $T \in L_p^q(G)$, $\| T \|_{p,q}$ is the smallest real number K satisfying

$$\| Tg \|_q \leqq K \| g \|_p$$