§ 7. Applications to divergence of Fourier series.

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satisfying $1 \leqq p < 2 < q \leqq \infty$, the series (6.6) converges normally in $L_p^q(G)$ to T. Next, T is the limit in E of

$$S_r = \sum_{n=1}^{r} \omega_n T_{K_n}$$

as $r \to \infty$ and, since it is plain that supp $S_r \subseteq \Omega$ for every r, (ii) is easily derived. Finally, if $\hat{T}$ were a measure μ, it would necessarily be the case that supp $\mu \subseteq \bar{\Omega}$ and so, for every $n \in N$, one would have by (6.1) and (6.4)

$$\begin{aligned} f_n(T) &= | u_n * Tv_n(0) | = | \textstyle\int_\Gamma \hat{u}_n \hat{v}_n d\mu | \\ &\leqq | \mu | (\bar{\Omega}), \end{aligned}$$

which is finite since Ω is relatively compact. However, this plainly would entail $f^*(T) < \infty$, in conflict with (6.8), so that T cannot be a measure and (iii) is verified. This completes the proof.

6.4 Remark. Theorem 6.3 was proved by Hörmander ([14], Theorem 1.9) for $G = R^n$ and any given pair (p, q) satisfying $1 \leqq p < 2 < q \leqq \infty$, this result being extended to a general noncompact LCA G by Gaudry [5]. The argument given by Hörmander (loc. cit. Theorem 1.6 and the remark immediately following) for the case $G = R^n$ can also be extended to a general LCA G and shows that, if *either* $q \leqq 2$ *or* $p \geqq 2$, then every $T \in L_p^q(G)$ is such that $\hat{T}$ is a measure [and indeed a measure of the form $\psi\lambda_\Gamma$, where $\psi \in L_{loc}^2(\Gamma)$ if $q \leqq 2$ and $\psi \in L_{loc}^p(\Gamma)$ if $p \geqq 2$, and so $\psi \in L_{loc}^2(\Gamma)$ in either case]. Thus the hypotheses made in Theorem 6.3 about p and q are necessary for the validity of the conclusion.

Part 3: Applications to Fourier series

§ 7. *Applications to divergence of Fourier series.*

7.1 Throughout §§ 7-10, G will denote an infinite Hausdorff compact Abelian group with character group Γ, and λ_G the Haar measure on G, normalised so that $\lambda_G(G) = 1$. For any $f \in L^1(G)$, $\hat{f}$ will denote the Fourier transform of f; for any finite subset Δ of Γ,

$$S_\Delta f = \sum_{\gamma \in \Delta} \hat{f}(\gamma)\gamma \qquad (7.1)$$

is the Δ-partial sum of the Fourier series of f; and sp (f) will stand for the spectrum of f, i.e., for the support supp $\hat{f} = \{\gamma \in \Gamma : \hat{f}(\gamma) \neq 0\}$ of $\hat{f}$. The term "trigonometric polynomial" will frequently be abbreviated to "t.p.". In addition, Φ will denote the largest torsion subgroup of Γ ([7], (A.4)), and π the natural map of Γ onto Γ/Φ. If Δ denotes a subset of Γ, $[\Delta]$ will stand for the subgroup of Γ generated by Δ.

By a *(convergence) grouping* we shall mean a sequence $\mathscr{D} = (\Delta_j)_{j \in N} = (\Delta_j)$ of finite subsets Δ_j of Γ such that

$$\left.\begin{array}{l} \Delta_j \subseteq \Delta_{j+1} \quad (j \in N); \\[1em] \bigcup_{j=1}^{\infty} \Delta_j = \Gamma_0 \text{ is a subgroup of } \Gamma \text{, said to be} \\ \qquad \textit{covered by } \mathscr{D}; \\[1em] \text{for each } j \in N,\ \Delta_j = \Omega_j + \Lambda_j \text{, where } \Lambda_j \text{ is a} \\ \qquad \text{nonvoid finite subset of } \Phi \text{ and } \Omega_j \text{ is a finite} \\ \qquad \text{subset of } \Gamma \text{ such that } \pi \mid \Omega_j \text{ is 1-1.} \end{array}\right\} \tag{7.2}$$

[The first two conditions are natural enough in the context described in 7.3, but the third is less so and may well be pointless.] The grouping $\mathscr{D}$ is said to be of *infinite type* if and only if $\pi(\Gamma_0)$ is infinite.

7.2 Examples. (i) Let Γ_0 be any countable subgroup of Γ such that $\Gamma_0 \cap \Phi = \{0\}$; for example, $\Gamma_0 = \{n\gamma_0 : n \in Z\}$, where $\gamma_0 \in \Gamma \setminus \Phi$. Then a grouping $\mathscr{D}$ covering Γ_0 results whenever $\Lambda_j = \{0\}$ and $\Delta_j = \Omega_j$ for every $j \in N$, where $(\Omega_j)_{j \in N}$ is any increasing sequence of finite subsets of Γ_0 with union equal to Γ_0. This grouping is of infinite type if and only if Γ_0 is infinite.

(ii) If G is connected, and if Γ_0 is any countable subgroup of Γ, then ([10], 2.5.6 (c), 8.1.2 (a) and (b) and 8.1.6) Γ_0 is an ordered group isomorphic to a discrete subgroup of R. Assuming $\Gamma_0 \neq \{0\}$, Γ_0 has a smallest positive element γ_0 and $\Gamma_0 = \{n\gamma_0 : n \in Z\}$. A natural grouping $\mathscr{D}$ covering Γ_0 is that in which $\Lambda_j = \{0\}$ and

$$\Delta_j = \Omega_j = \{n\gamma_0 : n \in Z, |n| \leq j\}$$

for every $j \in N$; this grouping is of infinite type.

7.3 A grouping $\mathscr{D} = (\Delta_j)_{j \in N}$ will be thought of as specifying one of the many possible ways in which one may interpret the convergence of Fourier series of functions f on G satisfying $sp(f) \subseteq \Gamma_0$, namely, as convergence of the corresponding sequence of partial sums $(S_{\Delta_j} f)_{j \in N}$.

Indeed, the conditions (7.2) guarantee that $\lim_{j\to\infty} S_{\Delta_j} f = f$ for all sufficiently regular such functions f. However, our concern rests with the possibility of constructing continuous functions f on G satisfying

$$\mathrm{sp}\,(f) \subseteq \Gamma_0, \ \overline{\lim_{j\to\infty}}\ \mathrm{Re}\, S_{\Delta_j} f(0) = \infty. \tag{7.3}$$

It will appear that the possibilities exhibit a fairly clear dichotomy, depending largely upon whether G is or is not 0-dimensional.

In the first place, it will emerge in 7.6 that the construction principle of § 2, applied to the Banach space $E = C(G)$ of continuous complex valued functions on G [with norm $\|\cdot\|$ equal to the maximum modulus] and to sequences of gauges of the type

$$f \mapsto \mathrm{Re}\, S_\Delta f(0) = \mathrm{Re} \int_G D_\Delta f \, d\lambda_G, \tag{7.4}$$

where D_Δ stands for the "Dirichlet function"

$$D_\Delta = \sum_{\gamma\in\Delta} \bar{\gamma}, \tag{7.5}$$

shows that the problem hinges on the existence of groupings $\mathscr{D}$ for which

$$\rho_j = \| D_{\Delta_j} \|_1 = \int_G | D_{\Delta_j} | \, d\lambda_G \to \infty. \tag{7.6}$$

Accordingly, and in view of the fact ([7], (24.26)) that G is 0-dimensional if and only if Γ coincides with Φ, it emerges that the dichotomy referred to may be expressed in the following way.

7.4 Two cases arise, namely:

(i) G is not 0-dimensional (i.e., $\Phi \neq \Gamma$). Then (see Example 7.2 (i)) there exist groupings $\mathscr{D} = (\Delta_j)$ of infinite type; and, for any such grouping, one can construct (fairly explicitly, as described in 7.6) continuous functions f on G satisfying (7.3). In particular [cf. Example 7.2 (i)], if Γ_0 is any countably infinite subgroup of Γ satisfying $\Gamma_0 \cap \Phi = \{0\}$, and if $(\Delta_j)_{j\in N}$ is any increasing sequence of finite subsets of Γ_0 with union Γ_0, we can construct a continuous f on G satisfying (7.3).

(ii) G is 0-dimensional (i.e., $\Phi = \Gamma$). Then there exists no grouping of infinite type. However, given any countable subgroup Γ_0 of Γ, there are groupings $\mathscr{D} = (\Delta_j)$ covering Γ_0, in which $\Omega_j = \{0\}$ and $\Delta_j = \Lambda_j$ is a finite subgroup of Γ_0, and for which

$$f = \lim_{j\to\infty} S_{\Delta_j} f$$

uniformly on G for every continuous f satisfying sp $(f) \subseteq \Gamma_0$.

Case (i) will be dealt with in § 8, case (ii) in § 9. The groupings described in case (ii) prove to be exceptional in various ways; see 9.3.

7.5 REMARK. Perhaps it should be stressed here that, if Γ_0 is any infinite subgroup of Γ, there is no obstacle to constructing continuous functions f such that sp $(f) \subseteq \Gamma_0$ and finite subsets $\Delta_j \subseteq \Delta_{j+1}$ of Γ_0 for which

$$\lim_j S_{\Delta_j} f(0) = \infty.$$

[One has in fact only to construct a continuous f such that sp $(f) \subseteq \Gamma_0$ and $\sum_{\gamma \in \Gamma} | \hat{f}(\gamma) | = \infty$; it is then trivial that there exist finite subsets Δ of Γ_0 for which $| S_\Delta f(0) |$ is arbitrarily large, so that we can choose a sequence (Δ_j) for which $\Delta_j \subseteq \Delta_{j+1}$ and $| S_{\Delta_j} f(0) | \to \infty$ with j.] However, the sets Δ_j obtained this way will not [and, in view of 7.4 (ii), cannot] in general be such that $\bigcup_{j=1}^{\infty} \Delta_j = \Gamma_0$. For more details, see A.5.1 and A.5.2 of the Appendix.

7.6 Suppose one is given a grouping $\mathscr{D} = (\Delta_j)_{j \in N}$ covering Γ_0 and satisfying (7.6). As is described in § 10, one may construct polynomials $q_{p_j, \nu}$ in two indeterminates over the real field (ν being a suitable fixed integer not less than 36 and p_j any positive number not less than $\| D_{\Delta_j} \|_\infty$) such that, for suitable unimodular complex numbers ξ_j, the t.p.s

$$Q_j = \xi_j \left(1 + \frac{1}{\nu}\right)^{-1} q_{p_j, \nu} (D_{\Delta_j}, \bar{D}_{\Delta_j})$$

satisfy

$$\left.\begin{aligned} &\| Q_j \| \leqq 1, \; sp\,(Q_j) \subseteq [\Delta_j] \subseteq \Gamma_0, \\ &S_{\Delta_j} Q_j(0) = \textstyle\int_G D_{\Delta_j} Q_j \, d\lambda_G \text{ is real and } \geqq \tfrac{1}{2} \rho_j. \end{aligned}\right\} \tag{7.7}$$

In view of (7.2), (7.6) and (7.7), one may choose inductively a sequence $(j_n)_{n \in N}$ of positive integers so that

$$\left.\begin{aligned} &S_{\Delta_{j_n}} Q_{j_n}(0) \text{ is real and } > n^3, \\ &j_n < j_{n+1}, \; sp\,(Q_{j_n}) \subseteq \Gamma_0. \end{aligned}\right\} \tag{7.8}$$

Accordingly, the t.p.s

$$u_n = n^{-2} Q_{j_n}$$

satisfy the conditions

$$\left.\begin{array}{l} \mathrm{sp}\,(u_n) \subseteq \Gamma_0,\ \sum_{n=1}^{\infty} \| u_n \| < \infty \\ \\ S_{\Delta_{j_n}} u_n (0) \text{ is real and } > n. \end{array}\right\} \tag{7.9}$$

At this point the construction in § 2 will yield integers $0 < n_1 < n_2 < \ldots$ and specifiable sequences $(\gamma_p)_{p \in N}$ of positive numbers such that each function of the form

$$f = \sum_{p=1}^{\infty} \gamma_p u_{n_p}$$

is continuous and satisfies

$$sp\,(f) \subseteq \Gamma_0,\ \lim_{p \to \infty} \mathrm{Re}\, S_{\Delta_{j_{n_p}}} f(0) = \infty. \tag{7.10}$$

A fortiori, f satisfies (7.3).

We add here that, if the Δ_j are symmetric, the D_{Δ_j} are real-valued, and we may work throughout with real-valued functions, replacing $\mathrm{Re}\, S_{\Delta_j} f$ by $S_{\Delta_j} f$ everywhere.

§ 8. *Discussion of case (i) :* G *not* 0-*dimensional*

8.1 In this case $\Phi \neq \Gamma$, and we begin by considering a finite subset of Γ of the form

$$\Delta = \Omega + \Lambda, \tag{8.1}$$

where Ω and Λ are finite subsets of Γ such that $\pi \mid \Omega$ is 1-1 and $\varnothing \neq \Lambda \subseteq \Phi$. We aim to show that (for a suitable absolute constant $k > 0$)

$$\| D_\Delta \|_1 \geqq k \left(\frac{\log N}{\log \log N} \right)^{\frac{1}{4}}, \tag{8.2}$$

provided $N = \mid \Omega \mid$ (the cardinal number of Ω) is sufficiently large.

8.2 Proof of (8.2). Introduce H as the annihilator in G of Φ and identify in the usual way the dual of H with Γ / Φ. Likewise identify the dual of $K = G/H$ with Φ ([7], (24.11)).