§ 9. Discussion of case (ii) : G 0-dimensional

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In particular, taking $\Delta_j = \{n \in Z : 2^j \leqq n < 2^{j+1}\}$ it can be arranged that

$$\sum_{n \in Z} \frac{\pm \hat{f}(n)}{(\log(2+|n|))^{\alpha}}$$

diverges for any preassigned distribution of signs $\pm$ and any preassigned $\alpha < \frac{1}{4}$.

Of course, much stronger results are derivable by using random (and unspecifiable!) changes of sign, but there seems little hope of making this even remotely constructive.

§ 9. *Discussion of case (*ii*) :* G 0-*dimensional*

9.1 In this case there is ([7], (7.7)) a base of neighbourhoods of zero in G formed of compact open subgroups W. For each such W the annihilator $\Delta = W^{\circ}$ in Γ of W is a finite subgroup of Γ. Define

$$k_W = \lambda_G(W)^{-1} \times \text{characteristic function of } W. \tag{9.1}$$

Then k_W is continuous, $k_W \geqq 0$, $\int_G k_W \, d\lambda_G = 1$. The transform $\hat{k}_W$ of k_W is plainly equal to unity on Δ. On the other hand, since W is a subgroup, we have for $a \in W$ and $\gamma \in \Gamma$

$$\begin{aligned} \hat{k}_W(\gamma) &= \int_G k_W(x) \overline{\gamma(x)} \, d\lambda_G(x) = \int_G k_W(x+a) \overline{\gamma(x)} \, d\lambda_G(x) \\ &= \int_G k_W(y) \overline{\gamma(y-a)} \, d\lambda_G(y) \\ &= \gamma(a) \hat{k}_W(\gamma), \end{aligned}$$

which shows that $\hat{k}_W(\gamma) = 0$ if $\gamma \in \Gamma \setminus \Delta$. Thus $\hat{k}_W$ is the characteristic function of Δ, and so

$$k_W = D_{W^{\circ}}. \tag{9.2}$$

By (9.1) and (9.2), a routine argument shows that, if $1 \leqq p < \infty$ and $f \in L^p(G)$, then

$$f = \lim_W S_{W^{\circ}} f \tag{9.3}$$

in $L^p(G)$; and that (9.3) holds uniformly for any continuous f.

9.2 PROOF OF 7.4 (ii). If Γ_0 is any countably infinite subgroup of Γ we can choose a sequence W_j of compact open subgroups of G such that

$W_{j+1} \subseteq W_j$ and $\Gamma_0 \subseteq \bigcup_{j=1}^{\infty} W_j^\circ$, where W_j° is a finite subgroup of Γ and $W_j^\circ \subseteq W_{j+1}^\circ$. The $\Delta_j = W_j^\circ \cap \Gamma_0$ satisfy (7.2) and, from (9.3),

$$f = \lim_j S_{\Delta_j} f \tag{9.4}$$

uniformly for any continuous f with $\mathrm{sp}(f) \subseteq \Gamma_0$. This verifies the statements made in 7.4 (ii).

9.3 By using the results in [3], more can be said in case (ii) of 7.4; cf. [3], Theorem (2.9) and Example (4.8).

Let $f \in L^1(G)$ and let Γ_0 be any countable subgroup of Γ containing $\mathrm{sp}(f)$. Choose the W_j as in 9.2. Then, apart from the fact that (W_j) is not in general a base at 0 in G (they can be chosen to be so if and only if G is first countable), (W_j) is an open-compact D''-sequence ([3], p. 188). The proof of Theorem (2.5) of [3] is easily modified to show that

$$f(x) = \lim_{j \to \infty} S_{W_j^\circ} f(x) \tag{9.5}$$

holds for almost all $x \in G$. Moreover, Theorem (2.7) of [3] applies to show that the majorant function

$$S^* f(x) = \sup_{j \in N} | S_{W_j^\circ} f(x) | \tag{9.6}$$

satisfies the estimates

$$\| S^* f \|_p \leqq 2 (p (p-1)^{-1})^{\frac{1}{p}} \| f \|_p \quad (1 < p < \infty) \tag{9.7}$$

$$\| S^* f \|_1 \leqq 2 + 2 \int_G |f| \log^+ |f| \, d\lambda_G, \tag{9.8}$$

$$\| S^* f \|_p \leqq 2 (1-p)^{\frac{1}{p}} \| f \|_1 \quad (0 < p < 1). \tag{9.9}$$

In particular, the convergence in (9.5) is dominated whenever

$$|f| \log^+ |f| \in L^1(G).$$

A more immediate consequence of (9.1) and (9.2) is a strong version of localisability of the convergence of Fourier series: if $f \in L^1(G)$ vanishes a.e. on some neighbourhood of $x_0 \in G$, we can choose the W_j so that $S_{\Delta_j} f(x_0) = 0$ for every sufficiently large j. [A suitable choice of W_j may be made once for all, independent of f, if G is first countable.] Nothing similar is true for general G; see, for example, [11], Vol. II, pp. 304-305.