NEW EXTENSION OF HÖLDER'S INEQUALITY

A NEW EXTENSION OF HÖLDER'S INEQUALITY

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1. We prove

$$[\Sigma(AB \dots L)^{ab \dots l}]^{1/ab \dots l} \leqq [\Sigma A^{\alpha a^{\alpha}}]^{1/\alpha a^{\alpha}} . [\Sigma B^{\beta b^{\beta}}]^{1/\beta b^{\beta}} \cdots [\Sigma L^{\lambda l^{\lambda}}]^{1/\lambda l^{\lambda}} \quad (1)$$

Where $A, B, \dots L$ are sets of non-negative real numbers, all sets having the same number of members and $a, \alpha\ b, \beta \dots l, \lambda$ are positive real numbers with

$$1/\alpha + 1/\beta + + + 1/\lambda = 1/k.$$

The inequality holds, writing p for $(ab \dots l)$

(a) When $k = 1$ for all p

(b) When $k < 1$ if $p \geqq k^{1/(1-k)}$

When $k > 1$ if $p \leqq k^{1/(1-k)}$

There is equality if and only if

(a) every number in one of the sets $A, B, \dots L$ or all but one is zero, and in the latter case those which are positive have the same rank.
Or

(b) $a^{\alpha} = b^{\beta} = \dots = l^{\lambda}$ and the sets $A^{\alpha}, B^{\beta}, \dots L^{\lambda}$ are proportional, and $k = 1$ or, if $k \neq 1$, $p = k^{1/(1-k)}$.

A variant of the result (1) is given in paragraph 7.

2) A well known extension of Hölder's Inequality [1], [5] is, in our notation [2]

$$\Sigma[AB \dots L] \leqq [\Sigma A^{\alpha}]^{1/\alpha} . [\Sigma B^{\beta}]^{1/\beta} . \dots [\Sigma L^{\lambda}]^{1/\lambda} \quad (2)$$

where $\Sigma 1/\alpha = 1$, which is (1) for $k = 1$, $p = 1$.

Less commonly cited is a further extension by Jensen [3], [6] of 2 for $k < 1$, $p = 1$.

A rather different type of extension has recently been given by Daykin and Eliezer [7].

Our result extends Jensen's by covering the cases $k = 1$, all p and determining the restrictions on p when $k \neq 1$.

4) To prove our result we first transform the inequality (1) by writing p for $(ab \dots l)$ to obtain

$$\Sigma[AB \dots L] \leqslant [\Sigma A^{\alpha a^{\alpha}}]^{p/\alpha a^{a}} \dots [\Sigma L^{\lambda l^{\lambda}}]^{p/\lambda l^{\lambda}}$$

which holds when

$$\Sigma p/\alpha a^{\alpha} \geqq 1 \tag{3}$$

by Jensen's Theorem [3] and Hölder's Inequality [1].

Now if

$$\Sigma 1/\alpha = 1/k \tag{4}$$

$$\begin{aligned} \Sigma p/\alpha a^{\alpha} &= (p/k)\, \Sigma k/\alpha a^{\alpha} \\ &\geqslant (p/k) \,.\, 1/(a^{k} b^{k} \dots l^{l}) \\ &= p/kp^{k} \end{aligned}$$

by Arithmetic Mean $\geqq$ Geometric Mean [4].

$$\left.\begin{array}{r} \text{Thus 1 holds when } p/kp^{k} \geqslant 1. \\ \text{This is so when } k < 1 \text{ if } p \geqq k^{1/(1-k)} \\ \text{and when } k > 1 \text{ if } p \leqq k^{1/(1-k)} \end{array}\right\} \tag{6}$$

The conditions for equality are readily seen.

5) If the conditions (6) are not met, the inequality (1) may still hold, as it is possible for it to be true when $\Sigma p/\alpha a^{\alpha}$ is not $\geqq 1$, and there is no simple test to cover this case: our conditions, when met, assure that $\Sigma p/\alpha a^{\alpha} \geqq 1$.

6) Some special cases of interest are:

$$\begin{array}{lll} \text{If } p = 1 & 1 \text{ holds for any} & k \leqq 1. \\ \quad p < 1/e & 1 \text{ holds if} & k > 1. \\ \quad p > 1/e & 1 \text{ holds if} & k < 1. \end{array}$$

7. A variant of (1) is

$$\begin{aligned} &[(\mathscr{A}\mathscr{B} \dots \mathscr{L})^{p} - \Sigma(AB \dots L)^{p}]^{1/p} \\ &\geqslant [\mathscr{A}^{\alpha a^{\alpha}} - \Sigma A^{\alpha a^{\alpha}}]^{1/\alpha a^{\alpha}} \dots [\mathscr{L}^{\lambda l^{\lambda}} - \Sigma L^{\lambda l^{\lambda}}]^{1/\lambda l^{\lambda}} \end{aligned} \tag{7}$$

To obtain this we consider

$$\Sigma(AB \dots L)^{p} + (\mathscr{A}^{\alpha a^{\alpha}} - \Sigma A^{\alpha a^{\alpha}})^{p/\alpha a^{\alpha}} + +$$

which by (1) is

$$\leqq (\Sigma A^{\alpha a^{\alpha}} + \mathscr{A}^{\alpha a^{\alpha}} - \Sigma A^{\alpha a^{\alpha}})^{p/\alpha a^{\alpha}} \dots$$

which gives (7).

8. Analagous integral inequalities only exist [8] when $\Sigma 1/\alpha a^{\alpha} = 1/p$, and then (1) reduces to one form of Holder's Inequality for which the integral analogue is well known.

REFERENCES

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