# 15. Resist symbols

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"Contain" and "include" are almost always used as synonyms, often by the same people who carefully coach their students that $\in$ and $\subset$ are not the same thing at all. It is extremely unlikely that the interchangeable use of contain and include will lead to confusion. Still, some years ago I started an experiment, and I am still trying it: I have systematically and always, in spoken word and written, used "contain" for $\in$ and "include" for $\subset$. I don't say that I have proved anything by this, but I can report that (a) it is very easy to get used to, (b) it does no harm whatever, and (c) I don't think that anybody ever noticed it. I suspect, but that is not likely to be provable, that this kind of terminological consistency (with no fuss made about it) might nevertheless contribute to the reader's (and listener's) comfort.

Consistency, by the way, is a major virtue and its opposite is a cardinal sin in exposition. Consistency is important in language, in notation, in references, in typography—it is important everywhere, and its absence can cause anything from mild irritation to severe misinformation.

My advice about the use of words can be summed up as follows. (1) Avoid technical terms, and especially the creation of new ones, whenever possible. (2) Think hard about the new ones that you must create; consult Roget; and make them as appropriate as possible. (3) Use the old ones correctly and consistently, but with a minimum of obtrusive pedantry.

## 15. Resist symbols

Everything said about words applies, mutatis mutandis, to the even smaller units of mathematical writing, the mathematical symbols. The best notation is no notation; whenever it is possible to avoid the use of a complicated alphabetic apparatus, avoid it. A good attitude to the preparation of written mathematical exposition is to pretend that it is spoken. Pretend that you are explaining the subject to a friend on a long walk in the woods, with no paper available; fall back on symbolism only when it is really necessary.

A corollary to the principle that the less there is of notation the better it is, and in analogy with the principle of omitting irrelevant assumptions, avoid the use of irrelevant symbols. Example: "On a compact space every real-valued continuous function $f$ is bounded." What does the symbol "$f$" contribute to the clarity of that statement ? Another example: "If $0 \leqq lim_n\, \alpha_n{}^{1/n} = \rho \leqq 1$, then $lim\,_n\, \alpha_n = 0$." What does "$\rho$" contribute here? The answer is the same in both cases (nothing), but the reasons for the presence of the irrelevant symbols may be different. In the first case "$f$" may be just a nervous habit; in the second case "$\rho$" is probably a preparation for the proof. The nervous habit is easy to break. The other is harder, because it involves more work for the author. Without the "$\rho$" in the statement, the proof will take a half line longer; it will have to begin with something like "Write $\rho = lim_n\ \alpha_n{}^{1/n}$." The repetition (of "$lim_n\ \alpha_n{}^{1/n}$") is worth the trouble; both statement and proof read more easily and more naturally.

A showy way to say "use no superfluous letters" is to say "use no letter only once". What I am referring to here is what logicians would express by saying "leave no variable free". In the example above, the one about continuous functions, "$f$" was a free variable. The best way to eliminate that particular "$f$" is to omit it; an occasionally preferable alternative is to convert it from free to bound. Most mathematicians would do that by saying "If $f$ is a real-valued continuous function on a compact space, then $f$ is bounded." Some logicians would insist on pointing out that "$f$" is still free in the new sentence (twice), and technically they would be right. To make it bound, it would be necessary to insert "for all $f$" at some grammatically appropriate point, but the customary way mathematicians handle the problem is to refer (tacitly) to the (tacit) convention that every sentence is preceded by all the universal quantifiers that are needed to convert all its variables into bound ones.

The rule of never leaving a free variable in a sentence, like many of the rules I am stating, is sometimes better to break than to obey. The sentence, after all, is an arbitrary unit, and if you want a free "$f$" dangling in one sentence so that you may refer to it in a later sentence in, say, the same paragraph, I don't think you should necessarily be drummed out of the regiment. The rule is essentially sound, just the same, and while it may be bent sometimes, it does not deserve to be shattered into smithereens.

There are other symbolic logical hairs that can lead to obfuscation, or, at best, temporary bewilderment, unless they are carefully split. Suppose, for an example, that somewhere you have displayed the relation

$$(*) \qquad \int_0^1 |f(x)|^2 dx < \infty,$$

as, say, a theorem proved about some particular $f$. If, later, you run across another function $g$ with what looks like the same property, you should resist the temptation to say "$g$ also satisfies (*)". That's logical and alphabetical nonsense. Say instead "(*) remains satisfied if $f$ is replaced by $g$", or, better, give (*) a name (in this case it has a customary one) and say "$g$ also belongs to $L^2(0,1)$".

What about "inequality (*)", or "equation (7)", or "formula (iii)"; should all displays be labelled or numbered? My answer is no. Reason: just as you shouldn't mention irrelevant assumptions or name irrelevant concepts, you also shouldn't attach irrelevant labels. Some small part of the reader's attention is attracted to the label, and some small part of his mind will wonder why the label is there. If there is a reason, then the wonder serves a healthy purpose by way of preparation, with no fuss, for a future reference to the same idea; if there is no reason, then the attention and the wonder were wasted.

It's good to be stingy in the use of labels, but parsimony also can be carried to extremes. I do not recommend that you do what Dickson once did [2]. On p. 89 he says: "Then ... we have (1) ... "—but p. 89 is the beginning of a new chapter, and happens to contain no display at all, let alone one bearing the label (1). The display labelled (1) occurs on p. 90, overleaf, and I never thought of looking for it *there*. That trick gave me a helpless and bewildered five minutes. When I finally saw the light, I felt both stupid and cheated, and I have never forgiven Dickson.

One place where cumbersome notation quite often enters is in mathematical induction. Sometimes it is unavoidable. More often, however, I think that indicating the step from 1 to 2 and following it by an airy "and so on" is as rigorously unexceptionable as the detailed computation, and much more understandable and convincing. Similarly, a general statement about $n \times n$ matrices is frequently best proved not by the exhibition of many $a_{ij}$'s, accompanied by triples of dots laid out in rows and columns and diagonals, but by the proof of a typical (say $3 \times 3$) special case.

There is a pattern in all these injunctions about the avoidance of notation. The point is that the rigorous concept of a mathematical proof can be taught to a stupid computing machine in one way only, but to a human being endowed with geometric intuition, with daily increasing experience, and with the impatient inability to concentrate on repetitious detail for very long, that way is a bad way. Another illustration of this is a proof that consists of a chain of expressions separated by equal signs. Such a proof is easy to write. The author starts from the first equation, makes a natural substitution to get the second, collects terms, permutes, inserts and immediately cancels an inspired factor, and by steps such as these proceeds till he gets the last equation. This is, once again, coding, and the reader is forced not only to learn as he goes, but, at the same time, to decode as he goes. The double effort is needless. By spending another ten minutes writing a carefully worded paragraph, the author can save each of his readers half an hour and a lot of confusion. The paragraph should be a recipe for action, to replace the unhelpful code that merely reports the results of the act and leaves the reader to guess how they were obtained. The paragraph would say something like this: "For the proof, first substitute $p$ for $q$, then collect terms, permute the factors, and, finally, insert and cancel a factor $r$."

A familiar trick of bad teaching is to begin a proof by saying: "Given $\varepsilon$, let $\delta$ be $\left(\dfrac{\varepsilon}{3M^2 + 2}\right)^{1/2}$". This is the traditional backward proof-writing of classical analysis. It has the advantage of being easily *verifiable* by a machine (as opposed to *understandable* by a human being), and it has the dubious advantage that something at the end comes out to be less than $\varepsilon$, instead of less than, say, $\left(\dfrac{(3M^2 + 7)\,\varepsilon}{24}\right)^{1/3}$. The way to make the human reader's task less demanding is obvious: write the proof forward. Start, as the author always starts, by putting something less than $\varepsilon$, and then do what needs to be done—multiply by $3M^2 + 7$ at the right time and divide by 24 later, etc., etc.—till you end up with what you end up with. Neither arrangement is elegant, but the forward one is graspable and rememberable.

## 16. Use symbols correctly

There is not much harm that can be done with non-alphabetical symbols, but there too consistency is good and so is the avoidance of individually unnoticed but collectively abrasive abuses. Thus, for instance, it is good to use a symbol so consistently that its verbal translation is always the same. It is good, but it is probably impossible; nonetheless it's a better aim than no aim at all. How are we to read "$\in$": as the verb phrase "is in" or as the preposition "in" ? Is it correct to say: "For $x \in A$, we have $x \in B$," or "If $x \in A$, then $x \in B$" ? I strongly prefer the latter (always read "$\in$" as "is in") and I doubly deplore the former (both usages occur in the same sentence). It's easy to write and it's easy to read "For $x$ in A, we have $x \in B$"; all dissonance and all even momentary ambiguity is avoided. The same is