

# 11. Generalizations and open problems

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discuss what happens in the case (B). For simplicity we shall assume that  $K$  is totally real.

There is an element  $\varphi$  of the Galois group such that the sets  $\{\alpha^{(1)}, \dots, \alpha^{(\mu)}\}$  and  $\{\varphi(\alpha^{(1)}), \dots, \varphi(\alpha^{(\mu)})\}$  are neither identical nor disjoint. We may assume without loss of generality that

$$\{\varphi(\alpha^{(1)}), \dots, \varphi(\alpha^{(\mu)})\} = \{\alpha^{(1)}, \dots, \alpha^{(l)}, \alpha^{(\mu+1)}, \dots, \alpha^{(2\mu-l)}\}.$$

Here  $1 \leq l \leq \mu - 1$ . The forms  $L^{(1)}, \dots, L^{(\mu)}$  have rank  $\rho$ , and hence also  $L^{(1)}, \dots, L^{(l)}, L^{(\mu+1)}, \dots, L^{(2\mu-l)}$  have rank  $\rho$ . Denote the rank of  $L^{(1)}, \dots, L^{(l)}$  by  $r_1$  and the rank of  $L^{(1)}, \dots, L^{(\mu)}, \dots, L^{(2\mu-l)}$  by  $r_2$ . It is easily seen that  $r_2 \leq 2\rho - r_1$ , i.e. that

$$r_1 + r_2 \leq 2\rho.$$

Since  $\mu$  was chosen as small as possible with (10.10), and since  $l \leq \mu - 1$ , we have  $l < r_1 q$ . The number  $2\mu - l$  of elements of  $L^{(1)}, \dots, L^{(\mu)}, \dots, L^{(2\mu-l)}$  satisfies  $2\mu - l \leq r_2 q$  by (10.9). Thus

$$2\mu = l + (2\mu - l) < r_1 q + r_2 q \leq 2\rho q,$$

which contradicts (10.10). Hence (B) is impossible if  $K$  is totally real.

We have in fact used the hypothesis that  $K$  is totally real, for in general  $L^{(1)}, \dots, L^{(l)}$  need not be a Symmetric System, and  $l < r_1 q$  need not hold. The situation is therefore somewhat more complicated if  $K$  is not totally real.

## 11. GENERALIZATIONS AND OPEN PROBLEMS

**11.1.** The theorems of §7 and §10 can almost certainly be generalized to include  $p$ -adic valuations. I understand that work on this question is being done now. ( $p$ -adic versions of the results of §2 were discussed in §4.5). Next, suppose that  $K$  is an algebraic number field and that  $\alpha_1, \dots, \alpha_l$  are algebraic numbers such that  $1, \alpha_1, \dots, \alpha_l$  are linearly independent over  $K$ . It is likely that *for every  $\delta > 0$  there are only finitely many  $l$ -tuples of elements  $\beta_1, \dots, \beta_l$  of  $K$  with*

$$(11.1) \quad |\alpha_i - \beta_i| < \mathcal{H}(\beta)^{-1 - (1/l) - \delta} \quad (i = 1, \dots, l),$$

where  $\mathcal{H}(\beta)$  is a suitably defined height of  $\beta = (\beta_1, \dots, \beta_l)$ . A possible definition for  $\mathcal{H}(\beta)$  is

$$\mathcal{H}(\beta) = \prod_v \max(1, \|\beta_1\|_v, \dots, \|\beta_l\|_v),$$

where  $v$  runs through the valuations of  $K$  and where  $\| \cdot \|_v$  is defined as in §4.5. In view of (4.9),  $\mathcal{H}(\beta)$  is almost the same as  $H_K(\beta)$  if  $l = 1$ , and hence a theorem on (11.1) would generalize Le Veque's Theorem 4A. One could try to obtain a still more general theorem which would contain both the  $p$ -adic case and the case of a number field  $K$ . Such a result was put forward as a conjecture by Lang (1962, Ch. 6).

**11.2.** We have already said in §2.2 that it would be desirable to replace the factor  $q^\delta$  in Roth's Theorem by something smaller, say by a power of  $\log q$ . The same is true of the generalizations of Roth's Theorem to simultaneous approximation.

The theorems of §2, 7 and 10 are non-effective. For approximation to a single algebraic number  $\alpha$  there are the effective results of Baker (see §5), but for simultaneous approximation there are only the relatively special effective theorems of Baker (1967a), Feldman (1970a, 1970b) and Osgood (1970).

**11.3.** The following questions also appear to be very difficult. Suppose  $(\alpha_1, \dots, \alpha_l)$  is a point of transcendence degree  $d < l$ . The theorems of §7 deal with the case when the point is algebraic, i.e. when  $d = 0$ . What can one say for other values of  $d$ ? Perron (1932) and Schmidt (1962) obtained results, of about the same level of sophistication as Liouville's Theorem, which can be used to show that certain given points have transcendence degree  $l$ .

A better question perhaps is how close rational points can come to a given algebraic variety. We may reformulate this question in a homogeneous setting. Let  $V$  be a homogeneous variety defined over the rationals (i.e. one defined by homogeneous polynomial equations with rational coefficients) in  $E^n$  with  $n \geq 2$ . For every  $\mathbf{x} \neq \mathbf{0}$  we put

$$\psi(V, \mathbf{x}) = \Delta(V, \mathbf{x}) |\mathbf{x}|^{-1}$$

where  $\Delta(V, \mathbf{x})$  is the distance from  $\mathbf{x}$  to  $V$ . It is clear that  $\psi(V, \lambda \mathbf{x}) = \psi(V, \mathbf{x})$ ; the function  $\psi(V, \mathbf{x})$  may be interpreted as the "angle" between  $V$  and the vector  $\mathbf{x}$ . We are interested in inequalities of the type

$$(11.2) \quad \psi(V, \mathbf{x}) < c |\mathbf{x}|^{-\omega}$$

where  $\mathbf{x}$  runs through the integer points. We saw in §6 that Theorems 6A and 6C had such an interpretation. The best value of  $\omega$  for which (11.2) has

infinitely many integer solutions can always be found if  $V$  is linear, i.e. is a subspace. For the non-linear case we have neither a good generalization of Dirichlet's Theorem nor anything like Roth's Theorem.

Suppose now that  $V$  is a hypersurface containing no integer point  $\mathbf{x} \neq \mathbf{0}$  and defined by the equation  $F(\mathbf{x}) = 0$  where  $F$  is a form of degree  $d$  with rational integer coefficients. For every integer point  $\mathbf{x} \neq \mathbf{0}$  we have  $|F(\mathbf{x})| \geq 1$ , and since  $|\frac{\partial}{\partial x_i} F(\mathbf{x})| \leq c_1 |\mathbf{x}|^{d-1}$  ( $i=1, \dots, n$ ), the distance from  $\mathbf{x}$  to  $V$  is  $\geq c_2 |\mathbf{x}|^{1-d}$ , which in turn implies that

$$\psi(V, \mathbf{x}) \geq c_3 |\mathbf{x}|^{-d},$$

where the constants depend only on  $V$ . This inequality may be interpreted as a generalization of Liouville's Theorem. Any improvement of this inequality, even though perhaps it may apply only to special classes of non-linear hypersurfaces, would be of great interest and would shed light on certain diophantine equations different from the equations with norm forms discussed in §10.

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