THE CONVERSE TO THE FOUR VERTEX THEOREM

Autor(en): **Gluck, Herman**

Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **17 (1971)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **23.05.2024**

Persistenter Link: https://doi.org/10.5169/seals-44584

THE CONVERSE TO THE FOUR VERTEX THEOREM

by Herman GLUCK

The Four Vertex Theorem of classical differential geometry states: *a simple closed curve in the plane has at least four vertices.* All curves are understood to be of class C^2, since we must talk about their curvature, and a *vertex* is a point at which the curvature function has a local maximum or minimum. This theorem was first proved by Mukhopadhyaya [7] in 1909, and has been reproved and reexamined many times since then. The same result for non-convex simple closed curves in the plane was proved by A. Kneser in 1912 and may be found in [6]; further information is in [1] and [2].

In this paper I will prove a converse to the Four Vertex Theorem. This converse is actually the one-dimensional case of a more extensive result on the existence of spheres in Euclidean space with preassigned strictly positive Gaussian curvature (the generalized Minkowski problem) which the reader may find in [4], and for which the present paper may serve as an introduction. We have here all of the essential ideas yet few of the technicalities of the higher dimensional analogue.

The Four Vertex Theorem must be sharpened slightly before we formulate and prove its converse. To see why this is necessary, consider the unit circle $S^1 = \{(x, y): x^2 + y^2 = 1\}$ in the plane R^2, and the function $k: S^1 \to R^1$ defined by $k(x, y) = 1 + y + |y|$. Could there exist an embedding $G: S^1 \to R^2$ which takes S^1 onto a simple closed curve M^1 such that the curvature of M^1 at the point $G(x, y)$ is $k(x, y)$ for all $(x, y) \in S^1$? Note that the function $k: S^1 \to R^1$ takes the constant value 1 on the semicircle $y \leqq 0$. At each point of this semicircle, k achieves a relative minimum, so technically M^1 would have infinitely many vertices. Therefore the Four Vertex Theorem in its casual formulation does not preclude the existence of such an embedding. Nevertheless, no such embedding exists.

It is the vertex counting procedure which needs to be sharpened. Suppose M^1 is a simple closed curve in the plane R^2, and that $k: M^1 \to R^1$ is its curvature function. Then we have the following possibilities.

(1) Perhaps k is a constant function.

(2) Perhaps k is not constant and there are two points, p_1 and p_2, on M^1

such that k is weakly monotonically increasing on each of the two arcs of M^1 running from p_1 to p_2. In this case we say that the function k has *one maximum and one minimum* on M^1.

(3) Perhaps neither (1) nor (2) is true. Then there are two relative maximum points of k on M^1 which are separated by two relative minimum points, such that the values of k at the two relative minimum points are strictly less than the values of k at the two relative maximum points. It is also possible to find four points p_1, p_2, p_3, p_4 in cyclic order around M^1 such that $k(p_1) = k(p_3) < k(p_2) = k(p_4)$. We cannot guarantee that these four are relative extrema of k, however. Case (3) is summarized by saying that k has *at least two maxima and two minima on* M^1.

Using this vertex counting procedure, we now reformulate the

FOUR VERTEX THEOREM. *The curvature function of a simple closed curve of class* C^2 *in the plane is either constant or else has at least two maxima and two minima.*

Note that this version immediately precludes the existence of the embedding G discussed above. Throughout this paper we deal only with curves of strictly positive curvature, and our main result is as follows.

CONVERSE TO THE FOUR VERTEX THEOREM. *Let* $k: S^1 \to R^1$ *be a continuous strictly positive function which is either constant or else has at least two maxima and two minima. Then there is an embedding* $G: S^1 \to R^2$ *taking* S^1 *onto a convex simple closed curve* M^1*, such that the curvature of* M^1 *at the point* $G(\varphi)$ *is* $k(\varphi)$ *for all* $\varphi \in S^1$.

Furthermore, if k *is of class* C^r, $r \geqq 0$, *then* G *is of class* C^{r+1} *and* M^1, *if reparametrized by arc length, is actually of class* C^{r+2}.

In section 1 below, I review some preliminary information about plane curves. In section 2, the converse to the Four Vertex Theorem is transformed into a purely topological theorem about normal vector fields on the unit circle S^1 in R^2, the proof of which is outlined briefly. In sections 3 through 5, I record the details of this proof.

1. *Plane curves*

We collect here some relevant information about plane curves. Let $x = x(s)$, $y = y(s)$ be a C^2 parametrization by arc length of a plane curve.

Then the tangent vector $(u(s), v(s)) = (x'(s), y'(s))$ to the curve is of unit length and satisfies the Frenet equations:

$$u'(s) = -k(s)v(s) \qquad v'(s) = k(s)u(s),$$

where $k(s)$ is the curvature function. If the function $k(s)$, of class C^r, $r \geqq 0$, is given in advance, then existence and uniqueness theorems for ordinary differential equations [5] tell us that there is up to rigid motions in the plane exactly one curve, of class C^{r+2}, with this preassigned curvature function. Such a curve may be found explicitly, as follows.

We must solve the above Frenet equations together with the initial conditions $(x(s_o), y(s_o)) = (x_o, y_o)$ and $(u(s_o), v(s_o)) = (u_o, v_o)$, a unit vector. Pick an angle θ_o such that $\cos\theta_o = u_o$ and $\sin\theta_o = v_o$. Then define the angle of inclination function $\theta(s) = \theta_o + \int_{s_o}^{s} k(\sigma)\,d\sigma$. Next let $u(s) = \cos\theta(s)$ and $v(s) = \sin\theta(s)$. Note that

$$u'(s) = -\theta'(s)\sin\theta(s) = -k(s)v(s)$$
$$v'(s) = \theta'(s)\cos\theta(s) = k(s)u(s).$$

Finally let $x(s) = x_o + \int_{s_o}^{s} u(\sigma)\,d\sigma$ and $y(s) = y_o + \int_{s_o}^{s} v(\sigma)\,d\sigma$, and we have the desired curve.

Suppose we are given a curvature function $k(s)$, how do we decide if " the " corresponding plane curve is closed, say of length L? In fact, it is easy to show that this curve will be closed if and only if the following conditions hold:

(1) $k(s)$ is periodic with period dividing L.

(2) $\int_{s_o}^{s_o+L} k(\sigma)\,d\sigma$ is an integral multiple of 2π.

(3) If we define $\theta(s) = \int_{s_o}^{s} k(\sigma)\,d\sigma$, then

$$\int_{s_o}^{s_o+L} \cos\theta(\sigma)\,d\sigma = 0 = \int_{s_o}^{s_o+L} \sin\theta(\sigma)\,d\sigma.$$

Next we ask, to what extent must these conditions be strengthened in order to guarantee that " the " corresponding closed curve is a simple closed curve of length L? This question is rather difficult to answer in general; let us assume that $k(s) \geqq 0$ for all s. Then " the " corresponding curve is a simple closed curve of length L if and only if conditions (1) and (3) above hold, together with

$$(2') \qquad \int_{s_o}^{s_o+L} k(\sigma)\,d\sigma = 2\pi .$$

This is an easy exercise for the reader.

Suppose furthermore that $k(s) > 0$ for all s. For simplicity, put $s_o = 0$, so that $\theta(s) = \int_0^s k(\sigma)\,d\sigma$. The map $\theta \colon R^1 \to R^1$ is an orientation preserving diffeomorphism, since $d\theta/ds = k(s) > 0$, and $\theta([0, L]) = [0, 2\pi]$. Inverting this diffeomorphism, we write $s = s(\theta)$, with $ds/d\theta = 1/k(s(\theta))$. If we denote the unit tangent vector to the curve by $\vec{T}(\theta) = (\cos\theta, \sin\theta)$, and if by abuse of language we write $k(\theta)$ instead of $k(s(\theta))$, then changing the variable of integration in condition (3) from s to θ yields

$$(3') \qquad \int_{\theta=0}^{2\pi} \frac{\vec{T}(\theta)}{k(\theta)}\,d\theta = 0 .$$

So now our curve is a simple closed curve of length L if and only if conditions (1), (2′) and (3′) hold.

In the following sections I will prefer to parametrize a convex simple closed curve by its unit outward normal vector, rather than by its unit tangent vector, so as to maintain an analogy with the higher dimensional version of this problem treated in [4]. If $\varphi(s)$ is the angle of inclination of the unit outward normal vector, then $\varphi(s) = \theta(s) - \pi/2$, and the normal vector is

$$\vec{N}(\varphi) = (\cos\varphi, \sin\varphi) = (\sin\theta, -\cos\theta) .$$

Continuing our abuse of notation, we now write $k(\varphi)$ instead of $k(s(\varphi))$. Then (3′) is equivalent to

$$(3'') \qquad \int_{\varphi=0}^{2\pi} \frac{\vec{N}(\varphi)}{k(\varphi)}\,d\varphi = 0 .$$

In order to collect the above information in a form suitable for our purposes, first recall the Gauss map. If M^1 is a smooth (at least C^1 when parametrized by arc length) curve in the plane, then the *Gauss map* $\gamma \colon M^1 \to S^1$ assigns to each point $p \in M^1$ one of the two unit normal vectors $\gamma(p)$ to M^1 at p, in such a way that $\gamma(p)$ varies continuously with p. If M^1 is a simple closed curve, for example, we let $\gamma(p)$ be the unit outward normal vector to M^1 at p. If M^1 is a convex simple closed curve in the plane, of class C^r, $r \geq 2$, whose curvature at every point is strictly positive, then $\gamma: M^1 \to S^1$ is a C^{r-1} diffeomorphism.

We now summarize as follows.

LEMMA 1.1. *Let $k: S^1 \to R^1$ be a continuous strictly positive function such that* $\int_{\varphi=0}^{2\pi} \frac{\vec{N}(\varphi)}{k(\varphi)} d\varphi = 0$. *Then there exists a convex simple closed curve $M^1 \subset R^2$, with Gauss map $\gamma: M^1 \to S^1$, such that the curvature of M^1 at $\gamma^{-1}(\varphi)$ is $k(\varphi)$ for all $\varphi \in S^1$, and this curve is unique up to translations.*

Furthermore, if k is of class C^r, $r \geqq 0$, then $\gamma^{-1}: S^1 \to M^1 \subset R^2$ is a C^{r+1} embedding and M^1, if reparametrized by arc length, is of class C^{r+2}.

Suppose now that we are given a continuous strictly positive function $k: S^1 \to R^1$ and asked to find an embedding $G: S^1 \to R^2$ taking S^1 onto a simple closed curve M^1 such that the curvature of M^1 at the point $G(\varphi)$ is $k(\varphi)$ for all $\varphi \in S^1$. The first thing to check, of course, is whether or not the integral

$$\int_{S^1} \frac{\vec{N}(\varphi)}{k(\varphi)} d\varphi = \int_{\varphi=0}^{2\pi} \frac{\vec{N}(\varphi)}{k(\varphi)} d\varphi$$

vanishes. If it does, then the preceding lemma supplies the curve M^1 and the embedding $G = \gamma^{-1}: S^1 \to M^1 \subset R^2$ to answer our question.

Suppose, however, that the integral does not vanish, but that nevertheless an embedding $G: S^1 \to R^2$ as required can somehow be found. Consider the diffeomorphism $h = \gamma G: S^1 \to S^1$. Note that the curvature of M^1 at the point $\gamma^{-1}(\varphi) = G h^{-1}(\varphi)$ is $kh^{-1}(\varphi)$. Hence by (3″) we must have

$$\int_{S^1} \frac{\vec{N}(\varphi)}{kh^{-1}(\varphi)} d\varphi = 0 .$$

Conversely, if we could find a diffeomorphism $h: S^1 \to S^1$ which makes the above integral vanish, then by Lemma 1.1 there would exist a convex simple closed curve $M^1 \subset R^2$ with Gauss map $\gamma: M^1 \to S^1$, such that the curvature of M^1 at $\gamma^{-1}(\varphi)$ is $kh^{-1}(\varphi)$ for all $\varphi \in S^1$. It follows that $G = \gamma^{-1} h$ is also an embedding of S^1 onto M^1 in R^2 such that the curvature of M^1 at $G(\varphi)$ is $k(\varphi)$ for all $\varphi \in S^1$. We may summarize as follows.

LEMMA 1.2. *Given a continuous strictly positive function $k: S^1 \to R^1$, then there exists an embedding $G: S^1 \to R^2$ taking S^1 onto a convex simple* *closed curve M^1 such that the curvature of M^1 at $G(\varphi)$ is $k(\varphi)$ for all $\varphi \in S^1$, if and only if there exists a diffeomorphism $h\colon S^1 \to S^1$ such that*

$$\int_{S^1} \frac{\vec{N}(\varphi)}{kh^{-1}(\varphi)}\, d\varphi = 0 .$$

2. *Normal vector fields on the circle*

In this section we transform the converse to the Four Vertex Theorem into a topological theorem about normal vector fields on the unit circle $S^1 \subset R^2$. The point $(x, y) = (\cos\varphi, \sin\varphi)$ on S^1 will often be referred to as φ for short. At the point φ, the unit outward normal vector to S^1 is $\vec{N}(\varphi) = (\cos\varphi, \sin\varphi)$. Let $f\colon S^1 \to R^1$ be a continuous, not necessarily positive, function. Then the vector field $f(\varphi)\vec{N}(\varphi)$, $\varphi \in S^1$, will be called a *normal vector field* on S^1. Let $h\colon S^1 \to S^1$ be a C^∞ diffeomorphism, diffeotopic (that is, differentiably isotopic) to the identity 1_{S^1}. At the same time that h slides a point φ of S^1 over to its image $h(\varphi)$, we may imagine the normal vector $f(\varphi)\vec{N}(\varphi)$ to S^1 at φ being dragged along with φ, its length remaining fixed during this process, until it becomes the normal vector $f(\varphi)\vec{N}(h\varphi)$ to S^1 at $h(\varphi)$. If we write $f(\varphi)\vec{N}(h\varphi) = fh^{-1}(h\varphi)\vec{N}(h\varphi)$, we see that h moves the normal vector field $f(\varphi)\vec{N}(\varphi)$ to the normal vector field $g(\varphi)\vec{N}(\varphi)$, where $g = fh^{-1}$. Under these circumstances, we will say that the two normal vector fields on S^1 are *deformations* of one another. Note that the diffeomorphism h is required to be of class C^∞, even though the normal vector fields may only be continuous. This is a simple way of guaranteeing that the deformed normal vector field will automatically be as smooth as the original one.

The integral of a continuous normal vector field $f(\varphi)\vec{N}(\varphi)$ on S^1 is the vector

$$\int_{S^1} f(\varphi)\vec{N}(\varphi)\, d\varphi = \left(\int_{\varphi=0}^{2\pi} f(\varphi)\cos\varphi\, d\varphi\,,\ \int_{\varphi=0}^{2\pi} f(\varphi)\sin\varphi\, d\varphi \right),$$

in R^2, and if the vector field is more or less random, so is its integral. Our problem is to decide whether or not a given normal vector field on S^1 can be deformed so as to make its integral over S^1 vanish.

THEOREM 2.1. *A continuous normal vector field $f(\varphi)\vec{N}(\varphi)$ on the unit circle $S^1 \subset R^2$ can be deformed so that, after the deformation, its integral* *over* S^1 *vanishes if and only if the function* $f\colon S^1 \to R^1$ *is either constant or else has at least two maxima and two minima.*

ADDENDUM 2.2. *Under the above circumstances, if the function* $f\colon S^1 \to R^1$ *has just one maximum and one minimum, then it is impossible to find even a homeomorphism* $h\colon S^1 \to S^1$ *which makes the integral* $\int_{S^1} fh^{-1}(\varphi)\vec{N}(\varphi)\,d\varphi$ *vanish.*

The effect of these assertions on the Four Vertex Theorem and its converse is clear. We couple Theorem 2.1 with Lemmas 1.1 and 1.2 by considering the normal vector field $\dfrac{\vec{N}(\varphi)}{k(\varphi)}$ on S^1. An immediate consequence is the Converse to the Four Vertex Theorem stated in the introduction. Similarly, coupling Addendum 2.2 with Lemma 1.2, we get yet another proof of the Four Vertex Theorem for convex curves of strictly positive curvature.

Addendum 2.2 is very easy to prove, and we do so now. We are given a normal vector field $f(\varphi)\,\vec{N}(\varphi)$ on S^1, where f has just one maximum and one minimum. Such a vector field is sketched in Figure 1.

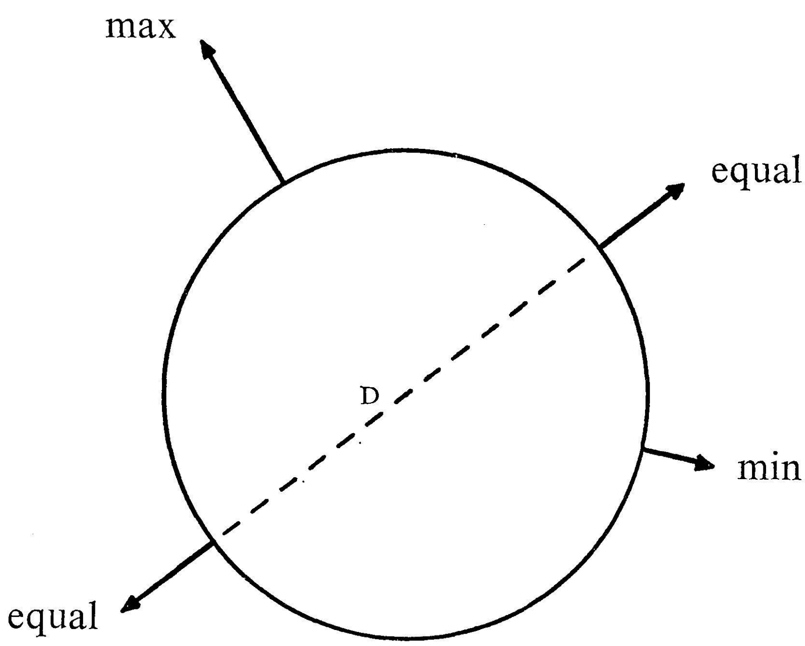

FIG. 1

By continuity there must be a diameter D of S^1 which separates the maximum and minimum points of f, such that at the ends of D the two values of f are equal. Then $f(\varphi)$ is always larger to one side of D than the other, so that $\int_{S^1} f(\varphi) \vec{N}(\varphi)\, d\varphi$ must have a nonzero component perpendicular to D. If h is any homeomorphism of S^1 onto itself, the function $fh^{-1}: S^1 \to R^1$ still has just one maximum and one minimum, so that by the same argument, $\int_{S^1} fh^{-1}(\varphi) \vec{N}(\varphi)\, d\varphi \neq 0$. This proves the Addendum, and therefore yields a fast proof of the Four Vertex Theorem for convex curves of strictly positive curvature.

To complete the proof of Theorem 2.1, we must start with a continuous function $f: S^1 \to R^1$ which is either constant or else has at least two maxima and two minima, and hunt for a diffeomorphism $h: S^1 \to S^1$, diffeotopic to the identity, such that $\int_{S^1} fh^{-1}(\varphi) \vec{N}(\varphi)\, d\varphi = 0$. The idea behind this hunt is most easily described by analogy with the usual topological proof of the Fundamental Theorem of Algebra [3, pp. 306-307].

The equation $\int_{S^1} fh^{-1}(\varphi) \vec{N}(\varphi)\, d\varphi = 0$ with unknown h contained in $\mathrm{Diff}(S^1)$, the group of diffeomorphisms of the circle, is the analogue of a complex polynomial equation with unknown z contained in the complex numbers. The map

$$I: \mathrm{Diff}(S^1) \to R^2,$$

$$I(h) = \int_{S^1} fh^{-1}(\varphi) \vec{N}(\varphi)\, d\varphi,$$

is continuous if $\mathrm{Diff}(S^1)$ is given the compact-open topology, hence remains continuous if $\mathrm{Diff}(S^1)$ is given any larger topology, such as the C^∞ topology [8, pp. 25-28]. The map I is the analogue of a complex polynomial function.

After some preliminary adjustments of f, which amount to replacing I by a similar map I_1 but do not affect the outcome of the problem, we restrict our hunt for h to a certain subgroup $\mathscr{D}$ of $\mathrm{Diff}(S^1)$. This subgroup is contractible, consists only of diffeomorphisms diffeotopic to the identity, and is the analogue of the complex plane. Inside $\mathscr{D}$ we construct a certain simple closed curve Σ^1, which is the analogue of a circle of large radius in the complex plane.

We approximate the adjusted function f by a step function g, and obtain a consequent approximation of I_1 by a map $J_1: \mathrm{Diff}(S^1) \to R^2$. This map J_1 plays the same role as that of the leading term of a complex polynomial function.

The argument draws to a close just as in the proof of the Fundamental Theorem of Algebra. We show that $I_1(\Sigma^1)$ misses the origin in R^2 and that $I_1 \mid \Sigma^1 : \Sigma^1 \to R^2 - \{0\}$ is an essential map by first establishing these results for J_1 and then capitalizing on the approximation between J_1 and I_1. Since Σ^1 is contractible within $\mathcal{D}$, the equation $I_1(h) = 0$ must have a solution within $\mathcal{D}$. We then easily produce a solution of $I(h) = 0$, and are done.

3. *Beginning of the proof of Theorem 2.1*

We are given a continuous normal vector field $f(\varphi) \vec{N}(\varphi)$ on the unit circle $S^1 \subset R^2$, such that the function $f: S^1 \to R^1$ is either constant or else has at least two maxima and two minima. If f is constant, then obviously $\int_{S^1} f(\varphi) \vec{N}(\varphi)\, d\varphi = 0$ by symmetry, so that no deformation is needed to make the integral vanish. Henceforth we may assume that f has at least two maxima and two minima.

By the remarks in the introduction, there are four points $\varphi_1{}^*$, $\varphi_2{}^*$, $\varphi_3{}^*$, $\varphi_4{}^*$ in counterclockwise order around S^1 such that

$$f(\varphi_1{}^*) = f(\varphi_3{}^*) = \mathfrak{N} < \mathfrak{M} = f(\varphi_2{}^*) = f(\varphi_4{}^*) .$$

Let $\zeta > 0$ be a small number, its actual size to be determined near the end of the proof. In order to make a later construction independent of the choice of ζ, we insist that $\zeta \leq \pi/2$. Let $E_1{}^*$ and $E_3{}^*$ be closed intervals about $\varphi_1{}^*$ and $\varphi_3{}^*$ such that

$$\mathfrak{N} - \zeta < f(\varphi^*) < \mathfrak{N} + \zeta \qquad \text{for} \qquad \varphi^* \in E_1{}^* \cup E_3{}^* .$$

Similarly let $D_2{}^*$ and $D_4{}^*$ be closed intervals about $\varphi_2{}^*$ and $\varphi_4{}^*$ such that

$$\mathfrak{M} - \zeta < f(\varphi^*) < \mathfrak{M} + \zeta \qquad \text{for} \qquad \varphi^* \in D_2{}^* \cup D_4{}^* .$$

We may take these four intervals to be disjoint.

Let $\varphi_1 = 0$, $\varphi_2 = \pi/2$, $\varphi_3 = \pi$ and $\varphi_4 = 3\pi/2$. We choose disjoint intervals E_1, D_2, E_3, D_4 about these four points, as follows. Let E_1 be the interval about φ_1 from $13\,\pi/8 + \zeta/4$ to $3\,\pi/8 - \zeta/4$. Let D_2 be the interval $[3\,\pi/8, 5\,\pi/8]$ about φ_2. Let E_3 be the interval $[5\,\pi/8 + \zeta/4, 11\,\pi/8 - \zeta/4]$ about φ_3, and finally let D_4 be the interval $[11\,\pi/8, 13\,\pi/8]$ about φ_4. See Figure 2.

Now let $h^*: S^1 \to S^1$ be an orientation preserving diffeomorphism (hence diffeotopic to the identity) taking φ_1, φ_2, φ_3, φ_4, E_1, D_2, E_3, D_4 to $\varphi_1{}^*$, $\varphi_2{}^*$, $\varphi_3{}^*$, $\varphi_4{}^*$, $E_1{}^*$, $D_2{}^*$, $E_3{}^*$, $D_4{}^*$, respectively. Then we have

$$f h^*(\varphi_1) = f h^*(\varphi_3) = \mathfrak{N} < \mathfrak{M} = f h^*(\varphi_2) = f h^*(\varphi_4),$$

and

$$\mathfrak{N} - \zeta < f h^*(\varphi) < \mathfrak{N} + \zeta \quad \text{for} \quad \varphi \in E_1 \cup E_3$$
$$\mathfrak{M} - \zeta < f h^*(\varphi) < \mathfrak{M} + \zeta \quad \text{for} \quad \varphi \in D_2 \cup D_4 .$$

Note that when the four closed intervals E_1, D_2, E_3 and D_4 are removed from S^1, the remainder A is the union of four open intervals, each of length $\zeta/4$.

The function $fh^*: S^1 \to R^1$ is best understood by comparison with the discontinuous step function $g: S^1 \to R^1$ defined by

$$g(\varphi) = \begin{cases} \mathfrak{N} & \text{for } \varphi \in D_2 \cup D_4 \\ \mathfrak{M} & \text{otherwise.} \end{cases}$$

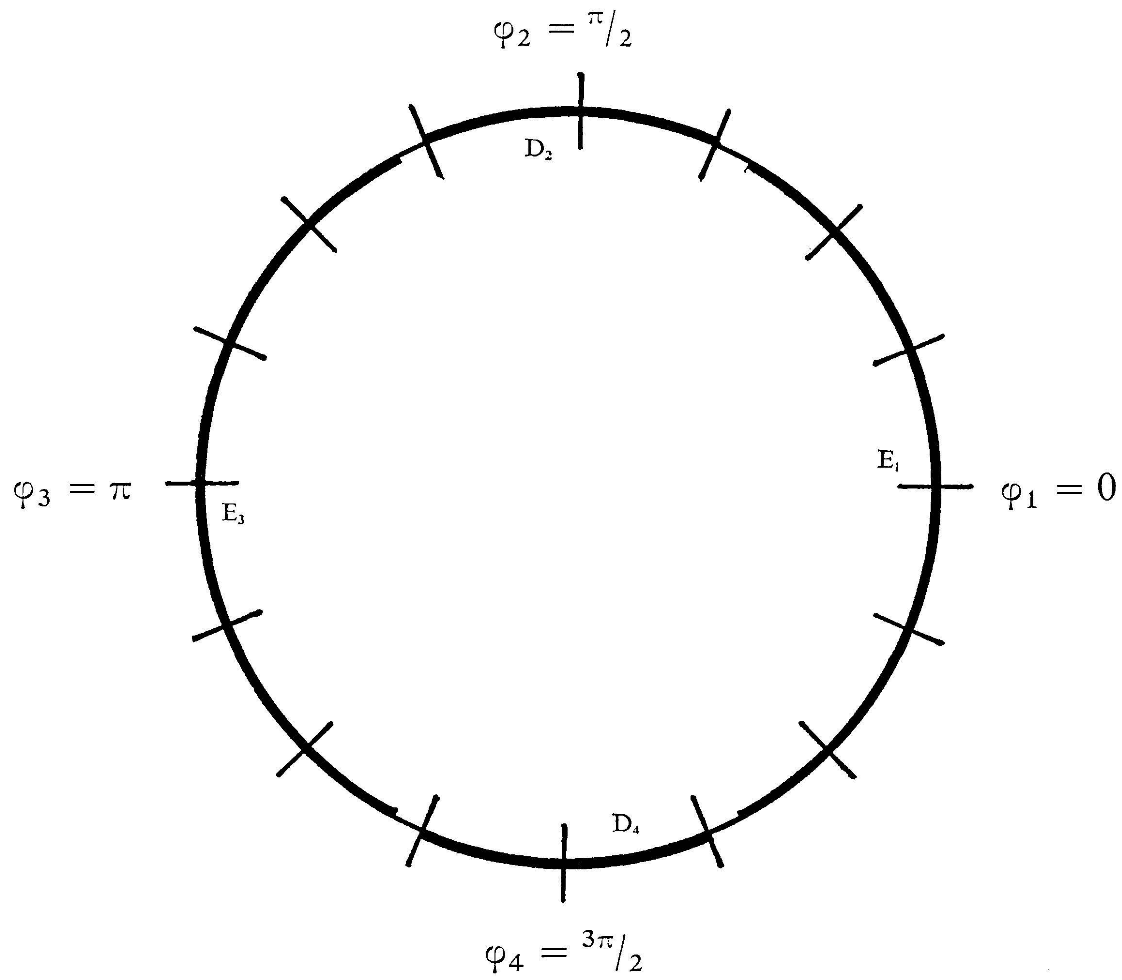

FIG. 2

Then fh^* and g are *ζ-approximations in measure* to one another on S^1, in the sense that $\left| fh^*(\varphi) - g(\varphi) \right| < \zeta$ on most of S^1 (namely, at least on $E_1 \cup D_2 \cup E_3 \cup D_4$), while the set X of points at which this inequality fails lies within A and has measure $< \zeta$.

We now replace f by fh^* in our original problem, and therefore consider in place of the map I, the map

$$I_1 \colon \mathrm{Diff}(S^1) \to R^2$$

$$I_1(h) = \int_{S^1} fh^* h^{-1}(\varphi) \vec{N}(\varphi) \, d\varphi \, .$$

If h is a zero of this map, then $h \, h^{*-1}$ will be a zero of the original map I, so that it suffices to find a zero of I_1.

Since ζ has yet to be determined, the reader should bear in mind the provisional character of the constructions in this section. The subsets $E_1{}^*$, $D_2{}^*$, $E_3{}^*$, $D_4{}^*$, E_1, E_3 and A of S^1, the diffeomorphism h^* of S^1 and the map $I_1 \colon \mathrm{Diff}(S^1) \to R^2$ all depend on the future choice of ζ. On the other hand, the step function $g \colon S^1 \to R^1$ is independent of ζ.

4. *The subgroup $\mathcal{D} \subset \mathrm{Diff}(S^1)$ and the simple closed curve $\Sigma^1 \subset \mathcal{D}$*

The subgroup $\mathcal{D}$ of $\mathrm{Diff}(S^1)$ will consist of those diffeomorphisms h of S^1 which restrict to the identity on some neighborhoods of $\varphi_1 = 0$ and $\varphi_3 = \pi$, these neighborhoods being allowed to vary with h. Such diffeomorphisms are orientation preserving and hence diffeotopic to the identity. Indeed, the linear isotopy

$$h_t(\varphi) = t \varphi + (1-t) \, h(\varphi) \, , \qquad 0 \leqq \varphi < 2\pi \, ,$$

shows that $\mathcal{D}$ is contractible to the identity 1_{S^1}. Within $\mathcal{D}$ we shall find a root of the equation $I_1(h) = 0$.

Next we construct the simple closed curve Σ^1 inside $\mathcal{D}$. To do this, it turns out to be convenient to first construct a certain 2-cell B^2 in $\mathcal{D}$ and then let Σ^1 be its boundary.

For each $(t, d) \in [-\pi/8, \pi/8] \times [1/2, 1]$ we will construct a diffeomorphism $h_{t,d} \in \mathcal{D}$. The action of $h_{t,d}$ on the two intervals $D_2 = [3\pi/8, 5\pi/8]$ and $D_4 = [11\pi/8, 13\pi/8]$ is what concerns us most. We insist that

$$h_{t,d}(\varphi) = \begin{cases} \pi/2 + d(\varphi - \pi/2) - t & \text{for} \quad \varphi \in D_2 \\ 3\pi/2 + (3/2 - d)(\varphi - 3\pi/2) + t & \text{for} \quad \varphi \in D_4 \, . \end{cases}$$

The behavior of $h_{t,d}$ on the rest of S^1 is not too important, except of course we want each $h_{t,d}$ to restrict to the identity on some neighborhoods of 0 and π, in order to guarantee that $h_{t,d} \in \mathcal{D}$. Note that $h_{t,d}$ does not stretch either D_2 or D_4, and it will be a matter of technical convenience later if we can assert the same for the four small intervals of length $\zeta/4$ which comprise

A. This is most easily done by extending the formula for $h_{t,d}$ on D_2 to the larger interval $[\pi/4, 3\,\pi/4]$, and similarly extending the formula from D_4 to $[5\,\pi/4, 7\,\pi/4]$. Since we insisted in section 3 that $\zeta \leqq \pi/2$, this enlargement of $D_2 \cup D_4$ swallows up A.

The extension of our partial definition of $h_{t,d}$ to one over all of S^1 is a straightforward but tiresome exercise. Instead one may appeal to a theorem of Palais [9], which guarantees that this extension is possible in such a way that the map

$$[-\pi/8, \pi/8] \times [1/2, 1] \to \mathscr{D} \subset \mathrm{Diff}(S^1)$$

$$(t, d) \to h_{t,d}$$

is continuous with respect to the C^∞ topology on $\mathrm{Diff}(S^1)$. This map is also one-to-one, as easily seen from the definition of $h_{t,d}$ on $D_2 \cup D_4$, and furthermore $\mathscr{D}$ is Hausdorff. Therefore the image of the map is a 2-cell B^2 in $\mathscr{D}$. The boundary of this 2-cell is the simple closed curve Σ^1 in $\mathscr{D}$ that we are looking for. Thus

$$\Sigma^1 = \{\, h_{t,d} \colon t = \pm\,\pi/8 \quad \text{while} \quad 1/2 \leqq d \leqq 1\,,$$

$$\text{or else}$$

$$-\,\pi/8 \leqq t \leqq \pi/8 \quad \text{while} \quad d = 1/2 \text{ or } 1\,\}\,.$$

The reader may find a verbal description of the diffeomorphisms $h_{t,d} \in \Sigma^1$ helpful. Suppose $0 \leqq t \leqq \pi/8$. Then $h_{t,1}$ acts on D_2 by sliding it rigidly to the right, t units. At the same time, $h_{t,1}$ acts on D_4 by instantaneously shrinking it from its original length of $\pi/4$ to a new length of $\pi/8$ and then sliding it t units to the right also.

As we travel around Σ^1 and t reaches $\pi/8$, then d begins to decrease from 1 to 1/2. As this happens, the image of D_2 under $h_{\pi/8,d}$ gradually shrinks from an interval of length $\pi/4$ to one of length $\pi/8$, its center remaining fixed at $3\,\pi/8$. At the same time, the image of D_4 under $h_{\pi/8,d}$ gradually grows from an interval of length $\pi/8$ to one of length $\pi/4$, its center remaining fixed at $13\,\pi/8$.

Continuing the trip around Σ^1, when d reaches 1/2 then t begins to decrease from $\pi/8$. As this happens, the image of D_2 under $h_{t,1/2}$ slides rigidly to the left, its length remaining shrunken to $\pi/8$. At the same time, the image of D_4 under $h_{t,1/2}$ also slides rigidly to the left, its length remaining full at $\pi/4$.

When we reach $h_{0,1/2}$ we have traveled half way around Σ^1. At this point the image of D_2 under $h_{0,1/2}$ is an interval of length $\pi/8$ centered at

$\pi/2$, while the image of D_4 is D_4 itself. The remaining half of the trip around Σ^1, corresponding to $-\pi/8 \leqq t \leqq 0$, is similar to the first half, except that the roles of D_2 and D_4 are interchanged, as are the words " left " and " right ".

The reader should note that the constructions in this section of the subsets $\mathscr{D}$, B^2 and Σ^1 of Diff (S^1), and therefore of the individual diffeomorphisms $h_{t,d}$, are independent of the future choice of ζ.

5. *Conclusion of the proof*

In this section we compute the image curve $I_1(\Sigma^1)$ in R^2, show that it misses the origin and finally show that it has winding number 1 about the origin. We do this by comparing I_1 with its " leading term ",

$$J_1 \colon \mathrm{Diff}(S^1) \to R^2$$

$$J_1(h) = \int_{S^1} g\, h^{-1}(\varphi) \vec{N}(\varphi)\, d\varphi ,$$

the step function $g\colon S^1 \to R^1$ having been defined in section 3. Note that

$$\begin{aligned} J_1(h) &= \int_{S^1} \mathfrak{N} \vec{N}(\varphi)\, d\varphi + \int_{h(D_2 \cup D_4)} (\mathfrak{M} - \mathfrak{N}) \vec{N}(\varphi)\, d\varphi \\ &= \int_{h(D_2 \cup D_4)} (\mathfrak{M} - \mathfrak{N}) \vec{N}(\varphi)\, d\varphi , \end{aligned}$$

since the first integral is 0 by symmetry. This expression for $J_1(h)$ shows that the discontinuity of g does not prevent J_1 from being a continuous map. Note that the map J_1 is independent of the choice of ζ, because this was true of the step function g. Since $\mathscr{D}$, B^2 and Σ^1 are independent of ζ, so are the restricted maps $J_1 \mid \mathscr{D}$, $J_1 \mid B^2$ and $J_1 \mid \Sigma^1$.

We now describe the image $J_1(B^2)$ by computing explicitly the vectors

$$J_1(h_{t,d}) = (\mathfrak{M} - \mathfrak{N}) \int_{S_{t,d}} \vec{N}(\varphi)\, d\varphi ,$$

where $S_{t,d} = h_{t,d}(D_2 \cup D_4)$ and $(t, d) \in [-\pi/8, \pi/8] \times [1/2, 1]$. We begin with a brief calculation:

$$\begin{aligned} \int_{\varphi=-\lambda/2}^{\lambda/2} \vec{N}(\varphi)\, d\varphi &= \int_{-\lambda/2}^{\lambda/2} (\cos\varphi, \sin\varphi)\, d\varphi = (\sin\varphi, -\cos\varphi) \Big|_{-\lambda/2}^{\lambda/2} \\ &= (2 \sin \lambda/2, 0) = 2 \sin \lambda/2\, \vec{N}(0) . \end{aligned}$$

By symmetry, we may state that in general the integral of $\vec{N}(\varphi)$ over an interval on S^1 of length λ centered at φ_o has the value $2 \sin \lambda/2\ \vec{N}(\varphi_o)$.

Next we note that $S_{t,d} = h_{t,d}(D_2 \cup D_4)$ consists of an interval of length $d\,\pi/4$ centered at $\pi/2 - t$ and another interval of length $(3/2-d)\,\pi/4$ centered at $3\,\pi/2 + t$. Using the preceding calculation, we find that

$$\begin{aligned} J_1(h_{t,d}) = (\mathfrak{M}-\mathfrak{N}) [2 \sin d\pi/8\ \vec{N}(\pi/2-t) \\ + 2 \sin (3/2-d)\,\pi/8\ \vec{N}(3\pi/2+t)] \,. \end{aligned}$$

If we let $d' = (4d-3)\,\pi/32$, then this expression simplifies to

$$J_1(h_{t,d}) = 4\,(\mathfrak{M}-\mathfrak{N})\ (\sin 3\,\pi/32 \cos d' \sin t,\ \cos 3\,\pi/32 \sin d' \cos t)\,.$$

One easily checks that J_1 embeds B^2 into the plane R^2 and takes the point $h_{0,3/4}$ to the origin. Furthermore, $J_1(h_{0,1})$ is one of the two points on $J_1(\Sigma^1)$ closest to the origin, the other being $J_1(h_{0,1/2})$. Its distance from the origin is

$$4\,(\mathfrak{M}-\mathfrak{N})\ \cos\ 3\pi/32\ \sin\ \pi/32 \approx .375\,(\mathfrak{M}-\mathfrak{N})\,.$$

Now we are in a position to pick ζ. We must do it so that the integrals $I_1(h_{t,d}) = \int_{S^1} fh^*\, h_{t,d}{}^{-1}(\varphi)\,\vec{N}(\varphi)\,d\varphi$ and $J_1(h_{t,d}) = \int_{S^1} g\, h_{t,d}{}^{-1}(\varphi)\,\vec{N}(\varphi)\,d\varphi$ are within $.375\,(\mathfrak{M}-\mathfrak{N})$ of each other for all $h_{t,d} \in \Sigma^1$.

Recall from section 3 that X denotes the set of points $\varphi \in S^1$ at which $|fh^*(\varphi) - g(\varphi)| \geqq \zeta$, and that X lies within A and has total measure $< \zeta$. Then $|fh^*\, h_{t,d}{}^{-1}(\varphi) - g\, h_{t,d}{}^{-1}(\varphi)| \geqq \zeta$ for precisely those φ which lie in $h_{t,d}(X)$. Since $h_{t,d}$ was constructed in section 4 not to stretch A, we know that $h_{t,d}(X)$ also has measure $< \zeta$. Hence $fh^*\, h_{t,d}{}^{-1}$ and $g\, h_{t,d}{}^{-1}$ are ζ-approximations in measure to one another on S^1.

Now we compute the distance from $I_1(h_{t,d})$ to $J_1(h_{t,d})$ for $h_{t,d} \in \Sigma^1$ (the same calculation is valid for $h_{t,d} \in B^2$, but is of no interest), as follows.

$$\begin{aligned} |I_1(h_{t,d}) - J_1(h_{t,d})| &= |\int_{S^1} f h^*\, h_{t,d}{}^{-1}(\varphi)\,\vec{N}(\varphi)\,d\varphi \\ &\quad - \int_{S^1} g\, h_{t,d}{}^{-1}(\varphi)\,\vec{N}(\varphi)\,d\varphi\,| \leqq \int_{S^1} |f h^*\, h_{t,d}{}^{-1}(\varphi) - g\, h_{t,d}{}^{-1}(\varphi)|\,d\varphi \\ &< \zeta\, 2\pi + (\mathfrak{M}-\mathfrak{N})\,\zeta = (2\pi + \mathfrak{M}-\mathfrak{N})\,\zeta\,. \end{aligned}$$

Since we want this to be less than $.375\,(\mathfrak{M}-\mathfrak{N})$, we may set it equal to $(\mathfrak{M}-\mathfrak{N})/3$, and therefore finally choose

$$\zeta = \frac{\mathfrak{M} - \mathfrak{N}}{3(2\pi + \mathfrak{M} - \mathfrak{N})}.$$

Note that $\zeta \leqq \pi/2$, as demanded at the beginning of section 3. Our previous constructions now lose their provisional character and become quite definite.

With this choice of ζ, $I_1(\Sigma^1)$ must also miss the origin, and is homotopic in $R^2 - \{0\}$ to $J_1(\Sigma^1)$, which links the origin once. Just as in the proof of the Fundamental Theorem of Algebra, it now follows that somewhere within the 2-cell B^2 there must exist a root h of the equation $I_1(h) = \int_{S^1} fh^* h^{-1}(\varphi) \vec{N}(\varphi) \, d\varphi = 0$. Then

$$I(h h^{*-1}) = \int_{S^1} f(h h^{*-1})^{-1} \vec{N}(\varphi) \, d\varphi = 0,$$

and our proof is over.

The author acknowledges partial support from the National Science Foundation via grant GP-19693.

BIBLIOGRAPHY

[1] Blaschke, W., *Vorlesungen über Differential Geometrie*, Berlin (1930); reprinted by Chelsea Publ. Co., New York (1967).

[2] Chern, S. S., " Curves and surfaces in Euclidean space ", Studies in Global Geometry and Analysis, vol. 4, *Math. Assoc. of Amer.* (1967), pp. 16-56.

[3] Eilenberg, S. and N. Steenrod, *Foundations of Algebraic Topology*, Princeton Univ. Press, Princeton (1952).

[4] Gluck, H., "The generalized Minkowski problem in differential geometry in the large", to appear in *Annals of Math.*

[5] Hurewicz, W., *Lectures on Ordinary Differential Equations*, Technology Press of MIT, Cambridge (1958).

[6] Jackson, S. B., " Vertices for plane curves ", *Bull. Amer. Math. Soc. 50* (1944), pp. 564-578.

[7] Mukhopadhyaya, S., " New methods in the geometry of a plane arc ", *Bull. Calcutta Math. Soc. 1* (1909), pp. 31-37.

[8] Munkres, J., *Elementary Differential Topology*, Princeton Univ. Press, Princeton (1963).

[9] Palais, R., " Local triviality of the restriction map for embeddings ", *Comm. Math. Helv. 34* (1960), pp. 305-312.

(Reçu le 26 août 1971)

Herman Gluck
Department of Mathematics
University of Pennsylvania
Philadelphia 19104